

(5.0) Definition. - Let G_1, G_2 be two groups.

A (set) map $f: G_1 \longrightarrow G_2$ is said to be a

group homomorphism if

$$\forall x, y \in G, \quad f(x \underset{\substack{\uparrow \\ \text{group operation of } G_1}}{*} y) = f(x) \underset{\substack{\uparrow \\ \text{gp. op. of } G_2}}{*} f(y)$$

Lemma. If $f: G_1 \longrightarrow G_2$ is a group homomorphism

then $f(e_1) = e_2$ (identity elements).

and $f(x)^{-1} = f(x^{-1}) \quad \forall x \in G_1.$

Proof. - Take $x = y = e_1$ in the definition, to get

$$f(e_1) = f(e_1)^2 \Rightarrow f(e_1) = e_2.$$

Now take $x = x$ and $y = x^{-1}$ in the definition, to get:

$$\left. \begin{array}{l} e_2 = f(e_1) = f(x) \underset{\substack{\uparrow \\ \text{gp. op. of } G_2}}{*} \cancel{f(x)^{-1}} f(x^{-1}) \\ \text{"} \\ f(x^{-1} \underset{\substack{\uparrow \\ \text{gp. op. of } G_1}}{*} x) = f(x^{-1}) \underset{\substack{\uparrow \\ \text{gp. op. of } G_2}}{*} f(x) \end{array} \right\} \Rightarrow f(x^{-1}) = f(x)^{-1}.$$

□

A group homomorphism is called an isomorphism (2) ~~(6)~~

$$(f: G_1 \rightarrow G_2)$$

if there exists a group homomorphism $g: G_2 \rightarrow G_1$ such

that $f(g(z)) = z \quad \forall z \in G_2$

$$g(f(x)) = x \quad \forall x \in G_1.$$

[Ex. - This is same as saying f is a set bijection.]

$G_1 \cong G_2$ (read: G_1 is isomorphic to G_2 .)

if an isomorphism $f: G_1 \rightarrow G_2$ exists.

~~(5.4)~~ Example I. $\langle a \mid a^k = e \rangle \cong \mathbb{Z}/_k\mathbb{Z}$
(6.1)

Example II. For a normal subgroup $N \trianglelefteq G$,

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/N \quad \text{is a (surjective)} \\ \psi & & \downarrow \psi \\ g & \xrightarrow{\quad} & g \cdot N \quad \left(\text{sometimes also denoted by } \bar{g} \right) \end{array}$$

group homomorphism (sometimes called natural/canonical projection onto quotient group).

Example III. $H = \{e, (12)(34), (13)(24), (14)(23)\}$ ~~7~~ 3

is a normal subgroup of $G = S_4$.

[This is an abnormal example. For $n \geq 5$, such normal subgroups of S_n do not exist.]

Ex. S_4/H has 6 elements.

$$\frac{|S_4|}{|H|} = \frac{24}{4}$$

Show that $S_4/H \cong D_6 (\cong S_3)$.

Example IV. $G_1 = GL_2(\mathbb{R}) \xrightarrow{\quad} G_2 = \mathbb{R}_{\neq 0}$
 $\downarrow \quad \quad \quad \uparrow$
 $A \xrightarrow{\quad} \det(A)$
 group under multiplication

is a group homomorphism.

$$SL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\} \subset GL_2(\mathbb{R})$$

is a normal subgroup. Why:

$$\det(TAT^{-1}) = \det(A) \quad \forall T, A \in GL_2(\mathbb{R})$$

(6.2) Definition. - Let G_1, G_2 be two groups

and $f: G_1 \rightarrow G_2$ a group hom.

(kernel of f) $\text{Ker}(f) := \{x \in G_1 \mid f(x) = e_2\} \subseteq G_1$

id. of G_2

(image of f) $\text{Im}(f) := \{y \in G_2 \mid \text{there exists } x \in G_1 \text{ with } f(x) = y\} \subseteq G_2$

(= $\{f(x) \mid x \in G_1\}$)

Lemma. - (1) $\text{Ker}(f)$ is a normal subgroup of G_1 .

(2) $\text{Im}(f)$ is a subgroup of G_2 .

Proof. - (1) $e_1 \in \text{Ker}(f) (\Rightarrow \text{non-empty; i.e. } \text{Ker}(f) \neq \emptyset)$

$$f(x) = e_2 \Rightarrow f(x^{-1}y) = f(x)^{-1}f(y) = e_2$$

$$f(y) = e_2$$

So, $\forall x, y \in \text{Ker}(f); x^{-1}y \in \text{Ker}(f)$. Therefore, $\text{Ker}(f)$ is a subgroup.

Now $\forall x \in G_1, k \in \text{Ker}(f)$ we have

$$f(x k x^{-1}) = f(x) \boxed{f(k)} f(x)^{-1}$$

" e_2

$$= f(x) f(x)^{-1} = e_2 \Rightarrow x k x^{-1} \in \text{Ker}(f)$$

Hence, $\text{Ker}(f)$ is normal.

(2) - left as an (easy) exercise. □

(6.3) First isomorphism theorem.

Let $f: G_1 \longrightarrow G_2$ be a group hom.

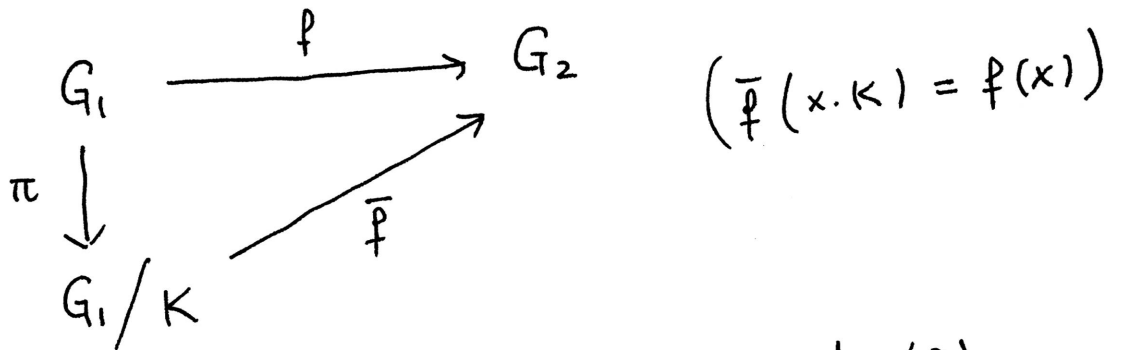
$K := \text{Ker}(f) \trianglelefteq G_1$ and $\pi: G_1 \longrightarrow G_1/K$

be the natural projection.

(1) There exists a unique $\bar{f}: G_1/K \longrightarrow G_2$

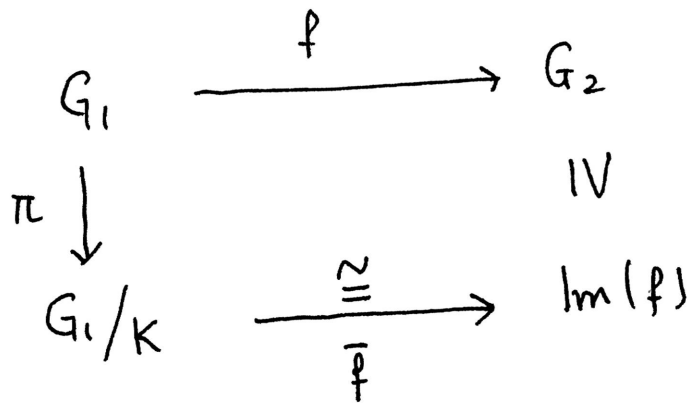
such that $f(x) = \bar{f}(\pi(x)) \quad \forall x \in G_1.$

Cartoon:



(2) \bar{f} sets up an iso. $G_1/K \cong \text{Im}(f)$

Cartoon:



(6.4) Easy lemma. $f: H_1 \longrightarrow H_2$ group hom.

(6)

(1) f is one-one (also called injective) iff (if and only if)

$$\text{Ker}(f) = \{e_1\}$$

(2) f is onto (also called surjective) if and only if

$$\text{Im}(f) = H_2$$

Proof. - (1) (\Rightarrow) Means: assume f is 1-1
prove $\text{Ker}(f) = \{e_1\}$

If $x \in \text{Ker}(f)$, then $f(x) = e_2 = f(e_1)$. As

f is 1-1, $f(x) = f(e_1) \Rightarrow x = e_1$. Hence $\text{Ker}(f) = \{e_1\}$

(\Leftarrow) Means: assume $\text{Ker}(f) = \{e_1\}$
prove f is 1-1

If $x, y \in H_1$ are such that $f(x) = f(y)$, then

$$f(x^{-1}y) = e_2 \Rightarrow x^{-1}y \in \text{Ker}(f) = \{e_1\}$$

$$\Rightarrow x^{-1}y = e_1 \Rightarrow x = y$$

(2) is much easier and left as an exercise.

□

(6.5) Proof of First Iso. Thm. (from (6.3) page 5).

(7)

$$(1) \quad \bar{f} : G_1/K \longrightarrow G_2$$

$$x \cdot K \longmapsto f(x)$$

seems to depend on a choice.

We have to make sure $xK = yK \Rightarrow f(x) = f(y)$

But $xK = yK$ means $x^{-1}y \in K = \text{Ker}(f)$

$$\Rightarrow f(x^{-1}y) = e_2 \Rightarrow f(x) = f(y) \checkmark$$

Check (easy) 1. \bar{f} is a group hom [Ex.]

Check 2. $\text{Ker}(\bar{f}) = \{e_i \cdot K\}$ (hence, by Lemma (6.4), \bar{f} is injective)

$$\bar{f}(xK) = e_2 \Rightarrow f(x) = e_2 \Rightarrow x \in K$$

hence $xK = e \cdot K$.

(2) By definition $\text{Im}(\bar{f}) = \text{Im}(f) \subset G_2$

$$\text{and } \left[\bar{f} : G_1/K \longrightarrow H_2 \right] \subset G_2 \quad \text{subgroup} \quad =: H_2$$

both injective & surjective

hence an iso.