(6.8) Definition. – Let G_1, G_2 be two groups.

A (set) map $f : G_1 \longrightarrow G_2$ is said to be a group homomorphism if

$$\forall\, x, y \in G, \quad f(x \underset{1}{*} y) = f(x) \underset{2}{*} f(y)$$

group operation of G_1 gp. op. of G_2.

Lemma. If $f : G_1 \longrightarrow G_2$ is a group homomorphism then $f(e_1) = e_2$ (identity elements).

and $f(x)^{-1} = f(x^{-1}) \quad \forall\, x \in G_1$.

Proof. – Take $x = y = e_1$ in the definition, to get

$$f(e_1) = f(e_1)^2 \Rightarrow f(e_1) = e_2.$$

Now take $x = x$ and $y = x^{-1}$ in the definition, to get

$$\left.\begin{array}{l} e_2 = f(e_1) = f(x) \underset{2}{*} \cancel{f(x)^{-1}}\, f(x^{-1}) \\ \| \\ f(x^{-1} \underset{1}{*} x) = f(x^{-1}) \underset{2}{*} f(x) \end{array}\right\} \Rightarrow f(x^{-1}) = f(x)^{-1}.$$

□

A group homomorphism $(f : G_1 \longrightarrow G_2)$ is called <u>an isomorphism</u>

if there exists a group homomorphism $g : G_2 \longrightarrow G_1$ such

that $f(g(z)) = z \quad \forall\ z \in G_2$

$g(f(x)) = x \quad \forall\ x \in G_1.$

[Ex. – This is same as saying f is a <u>set</u> bijection.]

$G_1 \cong G_2$ (read: G_1 is <u>isomorphic</u> to G_2.)

if an isomorphism $f : G_1 \longrightarrow G_2$ exists.

~~(5.4)~~ (6.1) Example I. $\langle a \mid a^k = e \rangle \cong \mathbb{Z}/_{k\mathbb{Z}}$

Example II. For a normal subgroup $N \trianglelefteq G$,

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/N \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ g & \longmapsto & g.N \end{array}$$

is a (surjective)

(sometimes also denoted by $\bar{g}$)

group homomorphism (sometimes called natural/ canonical projection onto quotient group).

Example III. $H = \{e, (12)(34), (13)(24), (14)(23)\}$

is a normal subgroup of $G = S_4$.

[This is an abnormal example. For $n \geq 5$, such normal subgroups of S_n do not exist.]

$$\frac{|S_4|}{|H|} = \frac{24}{4}$$

Ex. S_4/H has 6 elements.

Show that $S_4/H \cong D_6 \ (\cong S_3)$.

Example IV.

$$G_1 = GL_2(\mathbb{R}) \longrightarrow G_2 = \mathbb{R}_{\neq 0}$$

$$A \longmapsto \det(A)$$

group under multiplication

is a group homomorphism.

$$SL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\} \subset GL_2(\mathbb{R})$$

is a normal subgroup. Why:

$$\det(TAT^{-1}) = \det(A) \quad \forall\, T, A \in GL_2(\mathbb{R})$$

(6.2) **Definition.** – Let G_1, G_2 be two groups and $f : G_1 \longrightarrow G_2$ a group hom.

$$\text{(kernel of } f\text{)} \quad Ker(f) := \{ x \in G_1 \mid f(x) = \underset{\text{id. of } G_2}{e_2} \} \subseteq G_1$$

$$\text{(image of } f\text{)} \quad Im(f) := \{ y \in G_2 \mid \text{there exists } x \in G_1 \text{ with } f(x) = y \} \subseteq G_2$$

$$(= \{ f(x) \mid x \in G_1 \})$$

Lemma. – (1) $Ker(f)$ is a normal subgroup of G_1.

(2) $Im(f)$ is a subgroup of G_2.

Proof. – (1) $e_1 \in Ker(f)$ ($\Rightarrow$ non-empty; i.e. $Ker(f) \neq \phi$)

$$\begin{matrix} f(x) = e_2 \\ f(y) = e_2 \end{matrix} \Rightarrow f(x^{-1}y) = f(x)^{-1} f(y) = e_2$$

So, $\forall\, x, y \in Ker(f)$: $x^{-1}y \in Ker(f)$. Therefore, $Ker(f)$ is a subgroup.

Now $\forall\, x \in G_1$, $k \in Ker(f)$ we have

$$f(x k x^{-1}) = f(x) \underset{e_2}{\underset{\shortparallel}{\boxed{f(k)}}} f(x)^{-1}$$

$$= f(x) f(x)^{-1} = e_2 \implies x k x^{-1} \in Ker(f)$$

Hence, $Ker(f)$ is normal.

(2) - left as an (easy) exercise. $\square$

(6.3) First isomorphism theorem.

Let $f : G_1 \longrightarrow G_2$ be a group hom.

$K := \mathrm{Ker}(f) \trianglelefteq G_1$ and $\pi : G_1 \longrightarrow G_1/K$

be the natural projection.

(1) There exists a unique $\bar{f} : G_1/K \longrightarrow G_2$

such that $f(x) = \bar{f}(\pi(x)) \ \forall x \in G_1$.

Cartoon:

$$\begin{tikzcd} G_1 \arrow[r, "f"] \arrow[d, "\pi"'] & G_2 \\ G_1/K \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

$(\bar{f}(x.K) = f(x))$

(2) $\bar{f}$ sets up an iso. $G_1/K \cong \mathrm{Im}(f)$

Cartoon:

$$\begin{tikzcd} G_1 \arrow[r, "f"] \arrow[d, "\pi"'] & G_2 \\ & \mathrm{IV} \\ G_1/K \arrow[r, "\cong", "\bar{f}"'] & \mathrm{Im}(f) \end{tikzcd}$$

(6.4) Easy lemma. $f : H_1 \longrightarrow H_2$ group hom.

(1) f is one-one (also called injective) iff (if and only if)

$$\mathrm{Ker}(f) = \{e_1\}$$

(2) f is onto (also called surjective) if and only if

$$\mathrm{Im}(f) = H_2$$

Proof.- (1) ($\Rightarrow$) Means: assume f is 1-1
prove $\mathrm{Ker}(f) = \{e_1\}$

If $x \in \mathrm{Ker}(f)$, then $f(x) = e_2 = f(e_1)$. As f is 1-1, $f(x) = f(e_1) \Rightarrow x = e_1$. Hence $\mathrm{Ker}(f) = \{e_1\}$

($\Leftarrow$) Means: assume $\mathrm{Ker}(f) = \{e_1\}$
prove f is 1-1

If $x, y \in H_1$ are such that $f(x) = f(y)$, then

$$f(x^{-1}y) = e_2 \Rightarrow x^{-1}y \in \mathrm{Ker}(f) = \{e_1\}$$

$$\Rightarrow x^{-1}y = e_1 \Rightarrow x = y$$

(2) is much easier and left as an exercise.

$\square$

(6.5) Proof of First Iso. Thm. (from (6.3) page 5).

(1) $\bar{f} : G_1/K \longrightarrow G_2$

$x . K \longmapsto f(x)$

seems to depend on a choice.

We have to make sure $xK = yK \Rightarrow f(x) = f(y)$

But $xK = yK$ means $x^{-1}y \in K = \mathrm{Ker}(f)$

$\Rightarrow f(x^{-1}y) = e_2 \Rightarrow f(x) = f(y)$ ✓

Check (easy) 1. $\bar{f}$ is a group hom [Ex.]

Check 2. $\mathrm{Ker}(\bar{f}) = \{e_1 K\}$ (hence, by Lemma (6.4) $\bar{f}$ is injective)

$\bar{f}(xK) = e_2 \Rightarrow f(x) = e_2 \Rightarrow x \in K$

hence $xK = e.K$.

(2) By definition $\mathrm{Im}(\bar{f}) = \mathrm{Im}(f) =: H_2 \subset G_2$

and $\boxed{\bar{f} : G_1/K \longrightarrow H_2} \subset G_2$ subgroup

both injective & surjective

hence an iso.