Lecture 7

(7.0) Recall : • a group hom. $f : G_1 \longrightarrow G_2$ is a (set) map such that $f(xy) = f(x)f(y) \ \forall \ x, y \in G_1$.

• $\mathrm{Ker}(f)$ (Kernel of f) $= \{x \in G_1 \mid f(x) = e_2\}$

identity/unit of G_2

• $\mathrm{Im}(f)$ (Image of f)

$= \{y \in G_2 \mid y = f(x) \text{ for some } x \in G_1\}$

First Iso Thm.

$$\boxed{G_1 / \mathrm{Ker}(f) \cong \mathrm{Im}(f)}$$

e.g. $G_1 = GL_2(\mathbb{R}) \xrightarrow{\det} \mathbb{R}_{\neq 0}$

surjective gp. hom

$$\mathrm{Ker}(\det) = \{X \in GL_2(\mathbb{R}) \mid \det(X) = 1\}$$
$$= SL_2(\mathbb{R})$$

Hence, by 1^{st} iso. thm., $GL_2(\mathbb{R}) / SL_2(\mathbb{R}) \cong \mathbb{R}_{\neq 0}$

(7.1) Some properties of kernel.

(1) $\mathrm{Ker}(f) \trianglelefteq G_1$ (normal subgroup)

(2) f is 1-1 (or injective · means the same thing,

i.e. $f(x) = f(x') \Rightarrow x = x'$); if and only if

$$\mathrm{Ker}(f) = \{e_1\} \quad \text{unit of } G_1.$$

(7.3) Another way to interpret 1^{st} iso· thm.
(or use)

Lemma. – Let $f : G_1 \longrightarrow G_2$ be a surjective group hom

and let $H \trianglelefteq G_1$ be such that $f(x) = e_2$ $\forall\, x \in H$,

(i.e. $H \subseteq \mathrm{Ker}(f)$).

Then $H = \mathrm{Ker}(f) \iff G_1/H \cong G_2$.
(if and only if) induced from f.

Proof. – ($\Rightarrow$) 1^{st} iso· thm.

($\Leftarrow$) Let $G_1 \xrightarrow{\ \pi\ } G_1/H$ be the

natural projection. Since

$$\begin{array}{ccc} G_1 & \xrightarrow{\ \pi\ } & G_1/H \\ & \searrow^{f} & \wr\| \\ & & G_2 \end{array}$$

we get, $\mathrm{Ker}(f) = \mathrm{Ker}(\pi) = H$. $\square$

(7.4) Example.

$$\begin{array}{ccc} G_1 = \mathrm{Free}(2) & \xrightarrow{\quad p \quad} & G_2 = \mathbb{Z}^2 \\ \| & & \\ \langle a, b \mid \text{no rel}^{n} \rangle & & \ni \\ \ni & & \\ w & \longmapsto & (\#\text{ of } a\text{'s}, \#\text{ of } b\text{'s}) \end{array}$$

(e.g. $p(a^2 b^2 a^{-7}) = p(a^{-5} b^2) = (-5, 2)$)

p is a surjective group hom – easy exercise.

Let H = subgroup of G_1 generated by $\{ x y x^{-1} y^{-1} \mid x, y \in \mathrm{Free}(2) \} \subset \mathrm{Ker}(p)$

Claim: H is normal in G_1.

If the claim is true, then we have

$$G_1/H \cong \mathbb{Z}^2 \Rightarrow H = \mathrm{Ker}(p)$$

(same reason as Lemma (7.3))

$$\begin{array}{ccc} G_1/H & & \\ \| & & \\ \langle a, b \mid ab = ba \rangle & & \mathbb{Z}^2 \\ \ni & & \ni \\ a^m b^n & \longleftrightarrow & (m, n) \end{array}$$

(7.5) Proof of the claim. - The statement is quite general:

Let G be any group and $H \leq G$ be the subgroup generated by $\{ x y x^{-1} y^{-1} \mid x, y \in G \}$

Then $H \trianglelefteq G$. (called commutator subgroup;

a $xyx^{-1}y^{-1} \leftarrow$ "commutator of x & y")

Proof. We need to check that for any $a \in G$ and $xyx^{-1}y^{-1} \in H$, the following is true:

$$a(xyx^{-1}y^{-1})a^{-1} \in H$$

But $a(xyx^{-1}y^{-1})a^{-1}$

$$= (axa^{-1})(aya^{-1})(axa^{-1})^{-1}(aya^{-1})^{-1} \in H \checkmark$$

Question*: In order to prove that a subgroup $H \leq G$ is <u>normal</u>, is it enough to show $\cancel{\forall a \in G, h \in H}$

$$aha^{-1} \in H \quad \forall \text{ \underline{generator} } h \text{ of } H \text{ ?}$$

(meaning, is it enough to check the statement for $h \in A \subset H$?
$\uparrow$
<u>a</u> set of generators)

Answer: Yes. Because

$$a(h_1 h_2 \ldots h_\ell)a^{-1} = (ah_1a^{-1})(ah_2a^{-1}) \cdots (ah_\ell a^{-1})$$

$\square$

Question - In order to prove that a subgroup $H \leq G$ is normal is it enough to check $aha^{-1} \in H$ for every <u>generator</u> $a \in G$?

Answer - Yes. Because

$$a_1 a_2 \dots a_\ell \, h \, (a_1 a_2 \dots a_\ell)^{-1}$$
$$= a_1 \left(a_2 \left(\cdots a_\ell h a_\ell^{-1} \cdots \right) a_2^{-1} \right) a_1^{-1} \qquad \square$$

Definition (7.6) Conjugation by $a \in G$

$$\mathrm{Conj}(a) : \begin{array}{ccc} G & \longrightarrow & G \\ \Psi & & \Psi \\ x & \longmapsto & axa^{-1} \end{array}$$ is a group hom. (& more)

[In representation theory, also denoted by $Ad(a)$ (adjoint)]

In fact $\quad \mathrm{Conj}(ab) = \mathrm{Conj}(a) \circ \mathrm{Conj}(b)$

composition of functions

$$\mathrm{Conj}(e) = \text{Identity } \begin{array}{ccc} G & \longrightarrow & G \\ x & \longrightarrow & x \end{array}$$

Hence $\left(\mathrm{conj}(a)^{-1} = \mathrm{conj}(a^{-1}) \right)$

inverse of a set map

$\boxed{\mathrm{Conj}(a) : G \to G \text{ is an } \underline{\underline{\text{iso}}}.}$

(7.7) Precise definition of a group given to us by generators and relations.

Let A be a set and $\mathcal{R} \subset \mathrm{Free}(A)$.

A (generators), $\mathcal{R}$ (relations)

$$\langle A \mid \mathcal{R} \rangle := \mathrm{Free}(A) \Big/ \left\{\begin{array}{l}\text{Smallest } \underline{\underline{\text{normal}}} \\ \text{subgroup of Free}(A) \\ \text{containing } \mathcal{R}. \text{ (call it } N_{\mathcal{R}}\text{)}.\end{array}\right.$$

Recall : $\mathrm{Free}(A) \ni w$ has a unique expression of the form $w = x_1^{n_1} x_2^{n_2} \cdots x_\ell^{n_\ell}$ where

$$\left\{\begin{array}{l} x_1, \ldots, x_\ell \in A \quad ; \quad n_1, n_2, \ldots, n_\ell \in \mathbb{Z}_{\neq 0} \, ; \\ x_1 \neq x_2 \, ; \, x_2 \neq x_3 , \ldots , x_{\ell-1} \neq x_\ell \end{array}\right.$$

(convention $\ell = 0$ correspond to empty word $= e \in \mathrm{Free}(A)$)

($w^{-1} = x_\ell^{-n_\ell} x_{\ell-1}^{-n_{\ell-1}} \cdots x_1^{-n_1}$)

For any group G and a subset $X \subset G$

$$\begin{array}{l}\text{Smallest normal subgroup} \\ \text{of } G \text{ containing } X\end{array} = \bigcap_{\substack{N \trianglelefteq G \\ X \subset N}} N$$

(Ex. intersection of normal subgps is normal)

Question. – What would it take to define a group hom. $f : \mathrm{Free}(A) \longrightarrow H$ (H : an arbitrary group)?

Answer. – • Specify $f(a) \in H$ for every $a \in A$. (and do it as you wish, because "Free = Nothing to check")

• $w \in \mathrm{Free}(A)$ $\quad w = x_1^{n_1} \cdots x_\ell^{n_\ell}$ $\underline{\underline{\text{uniquely}}}$

means $f(w) = f(x_1)^{n_1} \cdots f(x_\ell)^{n_\ell}$ is the only possible definition of $f(w) \in H$. (unambiguous!!)

<u>Moral</u>:
$$\left\{ \begin{array}{c} \text{Group homomorphisms} \\ \mathrm{Free}(A) \longrightarrow H \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Set maps} \\ A \longrightarrow H \end{array} \right\}$$

Question. – Let $G = \langle A \mid R \rangle$ as defined on the previous page. What would it take to define a group hom. $f : G \longrightarrow H$?

Answer. – • Specify $f(a) \in H$ $\forall a \in A$

~~• Make sure~~

$\rightsquigarrow$ $\tilde{f} : \mathrm{Free}(A) \longrightarrow H$ group hom.

• Make sure $\tilde{f}(r) = 0$ $\forall r \in R \subset \mathrm{Free}(A)$

(7.8) Example. Prove that there is a group hom.

$$f : D_{2n} \longrightarrow \{\pm 1\}$$

gp. w/ 2 element: $(-1)(-1) = +1$

which sends s_1, s_2 to -1.

Proof. $D_{2n} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s s_2)^n = e \rangle$

Means we have to make sure $\begin{cases} f(s_1) := -1 \\ f(s_2) := -1 \end{cases}$

preserves the relations

$$f(s_1)^2 = 1 = f(s_2)^2 = \cancel{f(s_1 s_2)^n}$$
$$= (f(s_1) f(s_2))^n$$

but this is clear. $\square$

$\mathrm{Ker}(f) = \{$ words in s_1, s_2 of <u>even</u> length $\}$

$$= \{ e, s_1 s_2, (s_1 s_2)^2, \ldots, (s_1 s_2)^{n-1} \} =: K$$

$$\cong \mathbb{Z}/n\mathbb{Z}$$

$[\mathrm{Im}(f) = \{\pm 1\}$ i.e. f is surjective$]$

Being kernel of a gp. hom., K is normal in D_{2n}

$$D_{2n}/K \cong \{\pm 1\} \text{ by } 1^{st} \text{ iso. thm.}$$

$\square$