Lecture 8

(8.0) Complete argument for "if H = subgroup of Free(2) generated by elements of the form $\alpha\beta\alpha^{-1}\beta^{-1}$, where $\alpha, \beta \in$ Free(2); then

$$H = \text{Kernel of } \text{Free}(2) \xrightarrow{p} \mathbb{Z}^2$$
$$w \longmapsto (\#a\text{'s in } w, \# b\text{'s in } w)$$
".

List of things we already know:

(1) H is normal in Free(2) (see (7.5) page 3).
[we used $w(aba^{-1}b^{-1}) = waw^{-1}.wbw^{-1}.(waw^{-1})^{-1}.(wbw^{-1})^{-1}$]

(2) set N = smallest normal subgroup of Free(2) containing $aba^{-1}b^{-1}$
(here and above, Free(2) = $\langle a, b \mid \text{no rel}^{\text{ns}} \rangle$)

We know $N \subseteq H$ and $H \subset \text{Ker}(p)$

since $H \trianglelefteq$ Free(2) & $aba^{-1}b^{-1} \in H$

(3) Ex. prove that $\langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2$
(hint: write maps both ways $a^k b^l \longleftrightarrow (k, l)$)

so $\text{Free}(2)/N \cong \mathbb{Z}^2 \cong \text{Free}(2)/\text{Ker}(p)$

$\Rightarrow N = H = \text{Ker}(p)$ $\square$

Observation: Smallest subgroup ~~gen. by~~ containing $x = aba^{-1}b^{-1} = \langle x \rangle$
$= \{x^n : n \in \mathbb{Z}\} \cong \mathbb{Z}$

Smallest <u>normal</u> subgp. containing x = $\text{Ker}(p)$ <u>NOT</u> even <u>finitely gen.</u>

(8.1) Back to the symmetric group.

S_n := permutations (bijections) $\{1,\dots,n\} \to \{1,\dots,n\}$

$s_i := (i\ \ i{+}1) \qquad (1 \le i \le n-1)$

We know: (1) $\{s_1, s_2, \dots, s_{n-1}\}$ generates S_n

(2) $s_i^2 = e$; $s_i s_j = s_j s_i$ if $|i-j| \ge 2$

$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \forall\ 1 \le i \le n-2$

(1) & (2) Mean: if we define

$$\mathcal{G}_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \,\middle|\, \begin{array}{l} \sigma_i^2 = e ;\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \ge 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle$$

then we have a $\underset{(1)}{\underline{\text{surjective}}}$ group $\underset{(2)}{\text{hom}}$

$$\pi_n = \text{the map } \begin{array}{ccc} \mathcal{G}_n & \longrightarrow & S_n \\ \Psi & & \Psi \\ \sigma_i & \longmapsto & s_i \end{array}.$$

We prove, by induction, that π_n is an iso. $\forall\ n \ge 2$

Base case $n = 2$, $\mathcal{G}_2 = \langle \sigma \mid \sigma^2 = e \rangle \cong \mathbb{Z}/2\mathbb{Z} \cong S_2 = \{e, (12)\}$

Induction hypothesis: we already know $\pi_1, \dots, \pi_n$ are isomorphisms.

Induction step: $\pi_{n+1} : \mathcal{G}_{n+1} \twoheadrightarrow S_n$ is surjective

$\mathcal{G}_{n+1} = \langle \sigma_1 \dots \sigma_n \mid$ same list of rel$^{\text{n}}$s as above$\rangle$

$\sigma_i \longmapsto (i\ \ i{+}1) = s_i$

so $\quad |\mathcal{G}_{n+1}| \geq (n+1)!$ (Note: we don't know, yet, if $\mathcal{G}_{n+1}$ is <u>even finite</u>!)

So, it is enough to show $|\mathcal{G}_{n+1}| \leq (n+1)!$. For this we will use induction.

Since $\{\sigma_1, \dots, \sigma_{n-1}\} \subset \mathcal{G}_{n+1}$ satisfy relations defining $\mathcal{G}_n$, we have a group homomorphism

$$i_n : \mathcal{G}_n \longrightarrow \mathcal{G}_{n+1}$$

(Note: we don't know yet, if it is injective!)

Let $G_n = \mathrm{Image}(i_n) \leq \mathcal{G}_{n+1}$

$\hookrightarrow$ subgroup of $\mathcal{G}_{n+1}$ generated by $\{\sigma_1, \dots, \sigma_{n-1}\}$

As $\mathcal{G}_n / \mathrm{Ker}(i_n) \cong G_n$, we know $|G_n| \leq n! = |\mathcal{G}_n|$

this is where induction is used!

We are going to prove: <u>Claim</u>: $\quad \left| \mathcal{G}_{n+1} / G_n \right| \leq n+1$

(a weak, but enough for our purposes)

Let us see why this claim implies what we wanted (i.e. $|\mathcal{G}_{n+1}| \leq (n+1)!$). This claim gives, by $|G/H| = |G|/|H|$,

$$|\mathcal{G}_{n+1}| \leq (n+1)\,|G_n| \leq (n+1) \cdot n! = (n+1)!$$

and hence $|\mathcal{G}_{n+1}| = (n+1)! \implies \pi_{n+1}$ is an iso

(8.2) Proof of the claim.

We identify some nice left cosets from $\mathcal{G}_{n+1}/G_n$

$H_{n+1} := e\cdot G_n \in \mathcal{G}_{n+1}/G_n$; $H_n = \sigma_n G_n$;

$H_{n-1} = \sigma_{n-1}\sigma_n G_n$; $\quad\ldots\quad$; $H_1 = \sigma_1\sigma_2\ldots\sigma_{n-1}\sigma_n G_n$

[How did I guess this? I took

H_ℓ = Inverse image under $\pi_{n+1} : \mathcal{G}_{n+1} \longrightarrow S_{n+1}$

of the subset $\{ w \in S_{n+1} \mid w(n+1) = \ell \}$

Ex. Prove this : $\forall\, x \in H_\ell$, $\quad \pi_{n+1}(x) : n+1 \mapsto \ell$]

This, in fact shows that $H_1, H_2, \ldots, H_{n+1}$ are all distinct. For, if $x \in H_k \cap H_\ell$ where $k \neq \ell$, then $\pi_{n+1}(x) \in \phi$ ($\pi_{n+1}(H_k)$ & $\pi_{n+1}(H_\ell)$ are clearly disjoint), which is absurd.

We are going to show that $\mathcal{G}_{n+1}/G_n = \{H_1, \ldots, H_{n+1}\}$

Here is a fun fact about coset space: let $H \leq G$ and $X \subset G/H$, $X \neq \phi$. If $gX = X$ for every $g \in G$

then $X = G/H$.

(This is obvious, in that even a single element, say $g\cdot H$, is enough to generate $h\cdot H = hg^{-1}(g\cdot H)$.]

So we are faced with: To prove: $\forall\ w \in \mathcal{G}_{n+1}$ and $1 \leq \ell \leq n+1$, $\quad w(H_\ell) \in \{H_1, \ldots, H_{n+1}\}$

It is enough to check this statement for $w = \sigma_k$ $(1 \leq k \leq n.)$

$\underline{\sigma_k \cdot H_\ell = ?}$ $\quad \sigma_k(\sigma_\ell \sigma_{\ell+1} \cdots \sigma_n H)$: $\quad$ (if $\ell \neq n+1$) $(1 \leq \ell \leq n)$

[$\ell = n+1$; $\quad H_{n+1} = G_n \implies \sigma_k H_{n+1} = \begin{cases} H_{n+1} & \text{if } 1 \leq k \leq n-1 \\ H_n & \text{if } k = n \end{cases}$

so we are good.]

Cases I. $\quad k < \ell - 1$: $\quad \sigma_k H_\ell = H_\ell$ because σ_k commutes with $\sigma_\ell, \sigma_{\ell+1}, \ldots, \sigma_n$ and $\sigma_k \in G_n$ $(k < \ell - 1 \leq n-1$

II. $\quad k = \ell - 1$: $\quad \sigma_{\ell-1} H_\ell = H_{\ell-1}$

III $\quad k = \ell$: $\quad \sigma_\ell H_\ell = H_{\ell+1}$

IV $\quad k \geq \ell + 1$:

$$\sigma_K H_\ell = \sigma_K(\sigma_\ell \sigma_{\ell+1} \cdots \sigma_{k-1} \sigma_K \cdots \sigma_n) \cancel{H}\, G_n$$

commutes with these

$$= \sigma_\ell \sigma_{\ell+1} \cdots \sigma_{k-2} \underbrace{(\sigma_k \sigma_{k-1} \sigma_k)}_{\substack{\parallel \\ \sigma_{k-1} \sigma_k \sigma_{k-1}}} \sigma_{k+1} \cdots \sigma_n \cdot G_n$$

commutes with all these

$= H_\ell$. $\quad$ So we are done! $\quad \square$

(8.3) How will we try to prove the existence of

sign : $S_n \longrightarrow \{\pm 1\}$ group hom. that sends

transposition every $(i\,j)$ to -1. ? More directly.

Given a permutation $w \in S_n$, we will write it as a product of transpositions, and compute its sign by computing the parity of the number of transpositions used.

$$\left(\begin{array}{l} \text{odd} \longmapsto -1 \\ \text{even} \longmapsto +1 \end{array}\right)$$

But how do we know the parity hasn't changed if we wrote w as a <u>different</u> product of (potentially different) transpositions?

This is one of the consequences of what we have proved!

Corollary: Let $w \in S_n$ and let

$$w = s_{i_1} \cdots s_{i_k} = s_{j_1} \cdots s_{j_\ell}$$

($i_1, \dots, i_k; j_1, \dots, j_\ell$ are indices from $\{1, \dots, n-1\}$)

Then $k \equiv \ell$ modulo 2.
(same parity)

(8.4) Beware of ~~the~~ a circular argument: here is a very quick and elegant, but wrong, argument to show that

$$\begin{aligned} \text{sign}: S_n &\longrightarrow \{\pm 1\} \text{ is a group hom.} \\ s_i &\longmapsto -1 \end{aligned}$$

Define

$$\begin{tikzcd} S_n \arrow[r, hook] & GL_n(\mathbb{R}) \\ \sigma \arrow[r, mapsto] & X_\sigma \end{tikzcd}$$

where

$$(X_\sigma)_{j,i} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{o/w} \end{cases}$$

Check (easy matrix mult)

$$X_{\sigma\tau} = X_\sigma X_\tau$$

$$(X_{\sigma\tau})_{ji} = 1 \iff j = \sigma(\tau(i)) \quad \checkmark$$

$$? \,\|$$

$$\sum_{k=1}^{n} (X_\sigma)_{jk} (X_\tau)_{ki} = 1 \iff \begin{array}{l} k = \tau(i) \\ j = \sigma(k) = \sigma(\tau(i)) \end{array}$$

Now

$$\begin{tikzcd} S_n \arrow[r, hook] \arrow[rr, bend right, "\text{sign}"'] & GL_n(\mathbb{R}) \arrow[r, "\det"] & \mathbb{R}_{\neq 0} \end{tikzcd}$$

sign = gp. hom. !! ?

Fun fact: $\det(XY) = \det(X).\det(Y)$ is proved by using that sign : $S_n \to \{\pm 1\}$ is a gp. hom.

In fact the two assertions are equivalent [Ex.]

(8.5) Second isomorphism thm

[Book calls it 3rd & 4th iso thms.]

Let G be a group and $N \trianglelefteq G$ a normal subgroup.

Consider the natural projection $\pi : G \longrightarrow G/N$.

Then : π sets up an <u>order / index preserving</u> bijection

Part 1.

$$\left\{ \begin{matrix} \text{subgroups } A \leq G \\ \text{s.t. } N \subset A \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{subgps } \pi(A) \\ \overline{A} \leq G/N \end{matrix} \right\}$$

$$\begin{matrix} \text{IU} & \qquad\qquad & \text{UI} \end{matrix}$$

$$\left\{ \begin{matrix} \text{Normal subgroups} \\ A \trianglelefteq G \text{ s.t.} \\ N \subset A \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{normal subgps} \\ \overline{A} \trianglelefteq G/N \end{matrix} \right\}$$

$$\left[\begin{matrix} A & \longmapsto & \pi(A) \\ & & \overline{A} \\ \pi^{-1}(\overline{A}) & \longleftarrow\!\shortmid & \end{matrix} \right]$$

The proof is routine once the bijections are given. It uses

following facts left as an exercise :

Ex. ① $f : G_1 \longrightarrow G_2$: group hom.

$$\boxed{\begin{matrix} H_1 \leq G_1 \Rightarrow f(H_1) \leq G_2 \\ H_2 \leq G_2 \Rightarrow f^{-1}(H_2) \leq G_1 \end{matrix}}$$

$$\boxed{\begin{matrix} N_1 \trianglelefteq G_1 \Rightarrow f(N_1) \trianglelefteq G_2 \\ \& \ f \text{ is surj} \end{matrix}}$$

$$\boxed{N_2 \trianglelefteq G_2 \Rightarrow f^{-1}(N_2) \trianglelefteq G_1}$$

② Image of f need not take normal subgps to normal subgroups in general. But it will if <u>f is surjective</u>.

(8.6) 2^{nd} iso. thm. - part 2.

$$G \xrightarrow{\pi} G/N$$

Let $H \trianglelefteq G$ be a normal subgroup such that $N \subseteq H$.

$\pi(H) = H/N \trianglelefteq G/N$. Then

$$G/H \cong (G/N)/(H/N)$$

Hint for a proof. - compose two gp. hom-s

$$G \xrightarrow{\pi} G/N \xrightarrow{\bar{\pi}} (G/N)/(H/N)$$

natural projection for the quotient group formed from the pair: $H/N \trianglelefteq G/N$

Being composition of two surjective maps, $G \to (G/N)/(H/N)$ is surjective. But its <u>kernel</u> is exactly $\underline{\pi^{-1}(H/N) = H}$ (check).

By 1^{st} iso. thm. then $G/H \cong (G/N)/(H/N)$

$\square$