Lecture 9

(9.0) So far we have learnt:

(i) Definitions of the terms: group, subgroup, subgroup generated by a subset, cyclic groups (1 generator), order of an element / group. Left/right cosets G/H, $H\backslash G$; normal subgroups, quotient groups, Group homomorphisms/isomorphisms; kernel / image of a gp. hom. Free group; Generators & relations.

(ii) Abstract results:

- $\text{Cyclic gps} = \{ \underset{\text{infinite cyclic}}{\mathbb{Z}} \; ; \; \underbrace{\mathbb{Z}/k\mathbb{Z} \; ; \; k \geq 1}_{\text{finite cyclic}} \}$

- $H \leq G \Rightarrow |H|$ divides $|G|$ (if these are finite numbers).

$$|G/H| = |G|/|H|$$

(in particular, $\text{ord}(a)$ divides $|G|$, $\forall a \in G$. For instance, if $|G| = p$: prime, $a \in G \setminus \{e\}$, implies $\text{ord}(a) = p$, i.e. $\{a\}$ generates G and $\cancel{a^p = e}$. $a^p = e$. Hence $G \cong \mathbb{Z}/p\mathbb{Z}$.)

- $G_1 \xrightarrow{f} G_2$ gp. hom. $\Rightarrow G_1/\text{Ker}(f) \cong \text{Im}(f)$

($\text{Ker}(f) \trianglelefteq G_1$ and $\text{Im}(f) \leq G_2$).

- $N \trianglelefteq G \Rightarrow \pi : G \longrightarrow G/N$ sets up an <u>inclusion</u> & <u>index</u> preserving bijection

$$\left\{ \begin{array}{c} \text{(normal) subgps of } G \\ \text{containing } N \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{(normal) subgps of} \\ G/N \end{array} \right\}$$

and $\forall\ H \trianglelefteq G$ st. $N \subset H$, we have

$$G/H \cong (G/N)/(H/N)$$

(iii) Concrete examples.

- D_{2n} = dihedral group of size $2n$
  = symmetries of a regular n-gon.

We have proved that D_{2n} admits the following two presentations.

$$D_{2n} = \langle s, r \mid s^2 = e = r^n ;\ srs = r^{-1} \rangle$$
$$= \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n = e \rangle$$

Let $H = \langle s_1 s_2 = r \rangle \leq D_{2n}$. Then H is normal in D_{2n};

$H \cong \mathbb{Z}/n\mathbb{Z}$ and $D_{2n}/H \cong \mathbb{Z}/2\mathbb{Z}$.

In fact H = Kernel of $\left(\begin{array}{ccc} D_{2n} & \xrightarrow{\text{sign}} & \{\pm 1\} \\ s_1, s_2 & \longmapsto & -1 \end{array} \right)$.

- $\text{Free}(2) = \langle a, b \mid \text{no rel}^{n} \rangle$.

Smallest normal subgp. containing $aba^{-1}b^{-1}$ = Commutator subgp

$= \text{Kernel} \left(\begin{array}{ccc} \text{Free}(2) & \xrightarrow{p} & \mathbb{Z}^2 \\ w & \longmapsto & (\#a\text{'s}, \#b\text{'s}) \text{ in } w \end{array} \right)$

[It is <u>NOT</u> finitely generated.]

- S_n = symmetric group on n letters. $\left(|S_n| = n! \right)$
  (= permutations of $\{1, \dots, n\}$)

- Cyclic notation: For $\{x_1, \dots, x_\ell\} \subset \{1, \dots, n\}$, the notation $\sigma = (x_1\ x_2\ \dots\ x_\ell) \in S_n$ represents the permutation

$$\sigma(y) = y \quad \forall\, y \notin \{x_1, \dots, x_\ell\}$$

$$\sigma(x_i) = \begin{cases} x_{i+1} & \forall\ 1 \le i \le \ell-1 \\ x_1 & \text{for } i = \ell \end{cases}$$

- Given $\pi \in S_n$, there is a unique way (up to trivial rearrangements of disjoint cycles) of writing π as a product of disjoint cycles. $\pi = \pi_1 \pi_2 \dots \pi_r$.

π_j is a cycle of length ℓ_j. $(1 \le j \le r)$

Cycle type of π = sequence of numbers $\ell_1, \ell_2, \dots, \ell_r$ written usually in a <u>non-increasing fashion</u>

[e.g. cycle type of $(142)(35)(678)$ = $3, 3, 2$.)

We also know $\operatorname{ord}(\pi) = \text{l.c.m.}\{\ell_1, \dots, \ell_r\}$

Ex: How many permutations are there ~~and~~ of a prescribed cycle type?

- Conjugation $\operatorname{Conj}(a) : G \longrightarrow G$, $x \longmapsto axa^{-1}$ is a group <u>iso</u>.

Let $\sigma \in S_n$ and $\pi = (x_1\ x_2\ \dots\ x_\ell) \in S_n$. Then

$$\sigma \pi \sigma^{-1} = (\sigma(x_1)\ \sigma(x_2)\ \dots\ \sigma(x_\ell))$$

- We know a presentation of S_n:

let $$\mathcal{G}_n := \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i^2 = e \ \forall i ; \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j| \ge 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right\rangle$$

Then $\mathcal{G}_n \cong S_n$. The crucial step in the proof

$$\sigma_i \longmapsto s_i = (i\ i{+}1)$$

was realizing that $\mathcal{G}_{n+1}/G_n = \{ \underbrace{G_n}_{H_{n+1}} ; \underbrace{\sigma_n G_n}_{H_n} ; \underbrace{\sigma_{n-1}\sigma_n G_n}_{H_{n-1}} ; \ldots ; \underbrace{\sigma_1\sigma_2\cdots\sigma_n G_n}_{H_1} \}$

where G_n = subgroup of $\mathcal{G}_{n+1}$ generated by $\{\sigma_1, \ldots, \sigma_{n-1}\}$.

Note: the aim of this argument was to show that $|\mathcal{G}_{n+1}| \leq (n+1)!$.

For this <u>we don't need to know if</u> <u>$H_k \cap H_\ell = \phi$ for $k \neq \ell$</u>

or H_k could be $= H_\ell$ for some $k \neq \ell$.

Since in any case, we can correctly conclude:

$|\mathcal{G}_{n+1}/G_n| \leq n+1$. (By Induction $|G_n| \leq n!$ hence the claim.).

(9.1) <u>Groups acting on sets</u>.

Let G be a group and X be a set. An <u>action of G on X</u> is a set map

$$G \times X \xrightarrow{\alpha} X$$
$$(g, x) \longmapsto \alpha(g, x)$$

(conveniently written as $g \cdot x$)

satisfying

(i) $e \cdot x = x \quad \forall\, x \in X$

(ii) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \quad \forall\, g_1, g_2 \in G$ and $x \in X$.

In other words we have <u>a group hom.</u> $G \longrightarrow \text{Bijections}\{X \to X\}$

$$g \longmapsto \alpha_g : x \longmapsto \alpha(g, x)$$

$$\left(\begin{array}{lcl} \alpha_e = \mathrm{Id}_X & \leftrightarrow & e.x = x \ \forall\, x \in X \\ \alpha_{g_1 g_2} = \alpha_{g_1} \circ \alpha_{g_2} & \leftrightarrow & (g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \ \forall\, g_1, g_2 \in G; x \in X \end{array} \right)$$

Given $\alpha : G \times X \to X$ (or equivalently, a gp. hom $G \longrightarrow \underbrace{\mathrm{Aut}_{\mathrm{Set}}(X)}$) we say <u>G acts on X</u>, sometimes denoted as $G \overset{\curvearrowright}{} X$
(α) if needs to be specified.

A word on notation:

$\mathrm{Aut}_{\mathrm{Set}}(X)$ = set of all bijections $X \to X$ + gp. operation = composition = symmetric gp. on X.

Often in mathematics Aut = short for automorphisms (means isomorphisms with self = auto)

subscript - a "structure"
(sets = no structure)

denotes such a "concrete group".

e.g. $G = GL_n(\mathbb{R}) \curvearrowright X = \mathbb{R}^n$ by

$$G \times X \xrightarrow{\alpha} X$$
$$\parallel$$
$$GL_n(\mathbb{R}) \times \mathbb{R}^n \xrightarrow[\text{matrix mult.}]{} \mathbb{R}^n$$

However, we want to highlight that $\forall\ g \in GL_n(\mathbb{R})$, the resulting map $\mathbb{R}^n \xrightarrow{g} \mathbb{R}^n$ is not just a set bijection, but preserves "vector space structure"

(i.e. $g(\alpha v_1 + \beta v_2) = \alpha\, g(v_1) + \beta\, g(v_2)$ $\forall\ \alpha, \beta \in \mathbb{R};\ v_1, v_2 \in \mathbb{R}^n$.)

Thus $GL_n(\mathbb{R}) = \mathrm{Aut}_{\mathbb{R}\text{-vs.}}(\mathbb{R}^n)$ in this new notation.

(9.2) Some more definitions in the context of

$G \circlearrowright X$ (other notations $\alpha : G \times X \to X$, simply written as $(g, x) \mapsto g \cdot x$.)

- <u>Orbit of $x \in X$</u> is the following <u>subset of X</u>:

$$G \cdot x := \{ g \cdot x \mid g \in G \} \subseteq X$$

- <u>Stabilizer of x</u> $\in X$ is the following <u>subgroup</u> of G:

$$\mathrm{Stab}_G(x) := \{ g \in G \text{ such that } g \cdot x = x \} \leq G$$

[Ex: Check $\mathrm{Stab}_G(x)$ is a subgroup of G.
<u>Not necessarily normal</u>.]

- <u>Fixed point set of $g \in G$</u> is the following subset of X:

$$X^g := \{ x \in X \mid g \cdot x = x \} \subseteq X$$

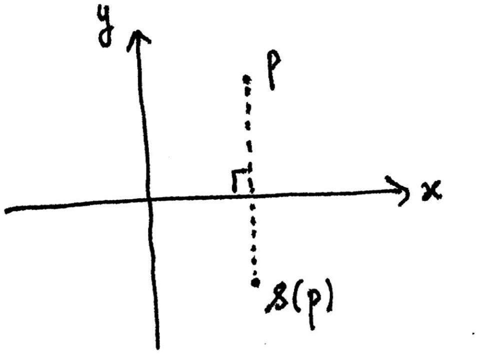
Ex. Consider the group hom.

$$\begin{aligned} D_{2n} &\longrightarrow GL_2(\mathbb{R}) \\ s &\longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ r &\longmapsto \begin{bmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{bmatrix} \end{aligned}$$

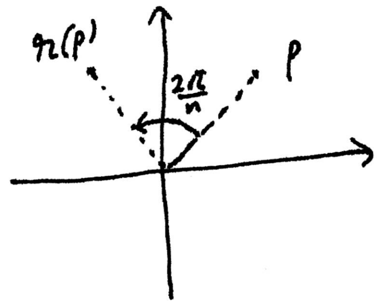
This defines $D_{2n} \circlearrowright \mathbb{R}^2$. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is clearly fixed by all elements of D_{2n}, so we omit it

$$X = \mathbb{R}^2 \setminus \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

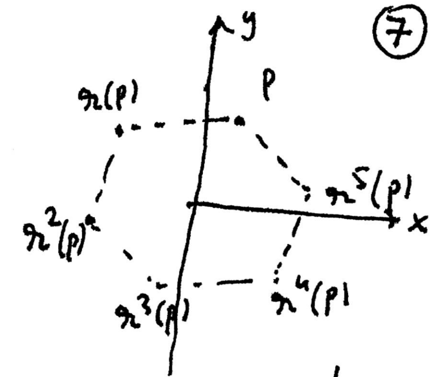
Let us find out orbits of points $p \in X$.



s = reflection about x-axis



r = rotation by $2\pi/n$



orbit of a point $p \in X = \mathbb{R}^2 \setminus 0$ under $\langle r \rangle$. $(n=6)$

all distinct

$$D_{2n} \cdot p \supset \{p, r(p), \ldots, r^{n-1}(p)\}$$

So $|D_{2n} \cdot p| = 2n$ unless $\underbrace{s(p) \in \{p, r(p), \ldots, r^{n-1}(p)\}}$

in this case (if $s(p) = r^k(p)$, $0 \le k \le n-1$)

$$|D_{2n} \cdot p| = n$$

Now $s(p) = r^k(p)$

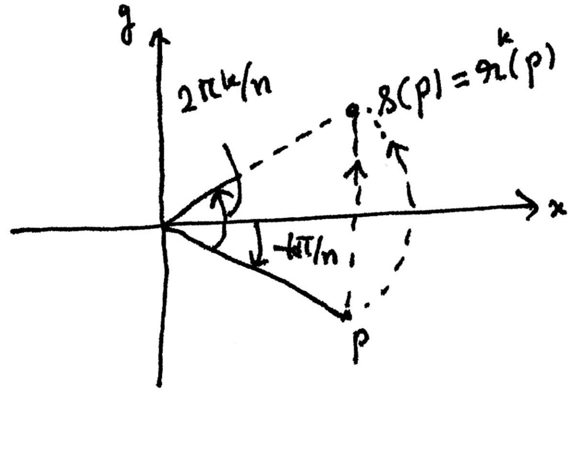
means – for p – reflection in X-axis $=$ rotation by $\frac{2\pi k}{n}$

So, $p = \begin{bmatrix} \cos\left(\frac{\pi k}{n}\right) \\ -\sin\left(\frac{\pi k}{n}\right) \end{bmatrix}$ (or any of its scalar multiples)

see picture

and $\operatorname{Stab}_{D_{2n}} \begin{bmatrix} \cos(\pi k/n) \\ -\sin(\pi k/n) \end{bmatrix} = \langle s r^k \rangle = \{ s r^k, (s r^k)^2 = e \}$

$$(s r^k)^2 = \boxed{s r^k s} r^k, \quad s r^k s = r^{-k}$$



(9.3) Let $G \circlearrowright X$.

(1) For every $x \in X$, we have a bijection

$$G/\mathrm{Stab}_G(x) \longrightarrow G.x$$

(2) For every $\sigma \in G$ and $x \in X$ we have iso. of groups

$$\begin{array}{ccc} \mathrm{Stab}_G(x) & \longrightarrow & \mathrm{Stab}(\sigma . x) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ g & \longmapsto & \sigma g \sigma^{-1} \end{array}$$

Proof. – (1) Define $\begin{array}{ccc} G & \longrightarrow & G.x \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ g & \longmapsto & g.x \end{array}$. By defn, this map is surjective. We notice that $g.x = h.x$ means $g^{-1}h \in \mathrm{Stab}_G(x)$ which is equivalent to $\cancel{\bar{g} = \bar{h}}$ $h \in g.\mathrm{Stab}_G(x)$.

Thus the induced map $G/\mathrm{Stab}_G(x) \longrightarrow G.x$ is both injective & surjective.

(2)
$$\begin{aligned} g \in \mathrm{Stab}_G(x) &\iff g.x = x \\ &\iff \sigma g \sigma^{-1}(\sigma x) = \sigma x \\ &\iff \sigma g \sigma^{-1} \in \mathrm{Stab}_G(\sigma(x)) \end{aligned}$$

Since $\mathrm{Conj}(\sigma)$ is an isomorphism of groups, we proved the required assertion. $\square$