Lecture 10

(10.0) Recall : G : a group $\qquad X$: a set

To have a "G-action on X" we need a set map $G \times X \to X$
$$(g, x) \mapsto g \cdot x$$

s.t.
$$\left.\begin{aligned} e \cdot x &= x \\ (g_1 g_2) \cdot x &= g_1 \cdot (g_2 \cdot x) \end{aligned}\right\} \forall \begin{aligned} g_1, g_2 &\in G \\ x &\in X \end{aligned}$$

(Or, equivalently, we need a gp. hom. $G \longrightarrow \mathrm{Aut}_{set}(X)$)

$\forall x \in X$ ⇝ Orbit of x, $G \cdot x \subset X$

⇝ Stabilizer of x, $\mathrm{Stab}_G(x) \leq G$

$$\boxed{G/\mathrm{Stab}_G(x) \xrightarrow{\text{bijection}} G \cdot x}$$

$$\boxed{\begin{aligned} \mathrm{Stab}(\sigma \cdot x) &\xleftarrow{\sim} \mathrm{Stab}(x) \\ \sigma g \sigma^{-1} &\longleftarrow g \end{aligned}}$$

(10.1) Three adjectives for group actions :

(1) Free. We say <u>G-action on X is free</u> if :
$$\begin{aligned} g \cdot x = x \quad &\Rightarrow \quad g = e. \\ (g \in G, x \in X) \end{aligned}$$

[Meaning : $\forall x \in X$, $\mathrm{Stab}_G(x) = \{e\}$. By <u>every orbit for a free action has same size $= |G|$</u>.]

(2) Transitive : We say G-action on X is <u>transitive</u> if $\forall x, y \in X$, $\exists g \in G$ such that $g \cdot x = y$.

In much simpler terms: G-action is transitive if and only if there is only one orbit.

(3) Faithful. G-action on X is *faithful* if $G \xrightarrow{\tau} \mathrm{Aut}_{\text{set}}(X)$ is 1-1. Meaning:

$$g \cdot x = x \ \forall x \in X \implies g = e.$$

(10.2) From yesterday's example:

$D_{2n} \circlearrowright \mathbb{R}^2 \setminus \{(0,0)\}$

Faithful ✓

Free × (there are orbits of size $n \neq |D_{2n}|$)

Transitive × (there are many orbits)

$S_n \circlearrowright \{1, 2, \ldots, n\}$

Faithful ✓

Free ×

Transitive ✓

$\left.\right\}$ There is only one orbit of size $n < |S_n|$ $(n \geq 3)$

(10.3) Counting how many orbits are there?

Burnside: $\boxed{\#\text{ of orbits} = \text{average } \#\text{ of fixed points}}$

Let $G\backslash X$ = set of orbits (since "G is on the left")

[an element of $G\backslash X \ni \mathcal{O}$ is a subset of X, of the form $G\cdot x$ – for <u>some</u> choice of $x \in \mathcal{O}$.]

Lemma. –

$$\frac{|X|}{|G|} = \sum_{\mathcal{O}\in G\backslash X} \frac{1}{|\mathrm{Stab}_G(x)|}$$

some element of $\mathcal{O}$.

Recall $\mathrm{Stab}(x) \cong \mathrm{Stab}(\sigma x)$ so every element of $\mathcal{O}$ has stabilizer of fixed size (depending only on $\mathcal{O}$)

Proof.

X = disjoint union of orbits

$$\Downarrow$$

$$|X| = \sum_{\mathcal{O}\in G\backslash X} |\mathcal{O}|$$

But $\mathcal{O} = G\cdot x \xleftrightarrow{\text{bijection}} G/\mathrm{Stab}(x)$

choose $x \in \mathcal{O}$

$$\Rightarrow \quad |X| = \sum_{\mathcal{O}\in G\backslash X} \frac{|G|}{|\mathrm{Stab}(x)|} \qquad \square$$

e.g. $S_n \circlearrowright X = \{1, \ldots, n\}$. There is only one orbit. Pick $n \in X$.

Then $\mathrm{Stab}_{S_n}(n) \cong S_{n-1}$ (as subgp. of S_n ~~gen. by~~)

So Lemma: $\dfrac{n}{n!} = \dfrac{1}{(n-1)!}$ ✓

(10.4) <u>Theorem (Burnside)</u>.

$$|G\backslash X| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

Proof. - Consider the following subset $F \subset G \times X$

$$F := \{(g,x)\in G\times X \mid g.x = x\}$$

I. $$|F| = \sum_{g\in G} |X^g|$$ $\quad |X^g| = \#\{x\in X \mid g.x = x\}$

fix g and count how many $x\in X$ are there so that $(g,x)\in F$ (i.e. $g.x = x$)

II. $$|F| = \sum_{x\in X}|Stab(x)|$$

fix x now and count how many g's.

but $|Stab(x)| = |Stab(\sigma x)|$, so

$$|F| = \sum_{\mathcal{O}\in G\backslash X} \underbrace{|\mathcal{O}|\cdot|Stab(x)|}_{=|G|}$$ $\quad x$: some $x\in\mathcal{O}$

$$= |G|\cdot|G\backslash X|$$ □

(10.5) Some examples. I.

$$\begin{array}{ccc} S_n & \circlearrowright & \{1,2,\dots,n\} \\ \Psi & & \parallel \\ \sigma & & X \end{array}$$

$|X^\sigma| = \#$ of 1-cycles in σ

(e.g. $n=5$, $X^{(123)(4)(5)} = \{4,5\}$)

X = disjoint union of orbits under σ

$\longleftrightarrow$ writing σ as product of disjoint cycles.

$$\mathrm{Stab}(k) = \{\pi \in S_n \mid \pi(k) = k\} \ni w$$

$$(1 \leq k \leq n) \qquad \cong \{\pi' \in S_n \mid \pi'(n) = n\} \ni (kn)w(kn)$$

(all $\cong S_{n-1} \leq S_n$.)

(10.6) A fun consequence of ~~Lemma (10.3) page 3.~~ (actually, just $G/\mathrm{stab}(x) \leftrightarrow G.x$)

Take, e.g., S_7. Pick a "decomposition" $7 = 3+2+2$.

X = set of all $\{1,\dots,7\} = P_1 \sqcup P_2 \sqcup P_3$

P_1 has 3 elts, P_2 has 2 elts, P_3 2 elts

$S_7 \circlearrowright X$ and this action is transitive (i.e. X is a single orbit).

$$\mathrm{Stab}((\{1,2,3\} \sqcup \{4,5\} \sqcup \{6,7\})) \cong S_3 \times S_2 \times S_2$$

$$\Rightarrow \quad |X| = \frac{|S_7|}{|\mathrm{Stab}(x_0)|} = \frac{7!}{3!\,2!\,2!}$$

$$x_0 = \{1,2,3\} \sqcup \{4,5\} \sqcup \{6,7\}$$

In general - multinomial coefficient -

$$\#\text{ partitions of } \left(\begin{array}{l} \{1,\dots,n\} = P_1 \sqcup P_2 \sqcup \dots \sqcup P_r \\ (\text{say, } |P_i| = \lambda_i \in \mathbb{Z}_{\geq 1} \text{ so that} \\ \lambda_1 + \dots + \lambda_r = n) \end{array} \right)$$

$$= \frac{n!}{\lambda_1!\,\lambda_2!\,\dots\,\lambda_r!}$$

(10.7) Another fun consequence (of Lemma (10.3) page 3.)

Prove that $\binom{p^r \cdot m}{p^r} \equiv m \pmod p$

where $p \in \mathbb{Z}_{\geq 2}$ is prime and $m \in \mathbb{Z}_{\geq 1}$.

Proof. Take $G = \mathbb{Z}/p^r\mathbb{Z}$, $X = \{x_1, \dots, x_m\}$ (some set with m elements)

E = set of all p^r elt. subsets of $G \times X$

$$|E| = \binom{p^r \cdot m}{p^r} .$$

$G \circlearrowright G \times X$ by $\sigma.(g, x) = (\sigma g, x)$

$\rightsquigarrow G \circlearrowright E$

$\{e_1, \dots, e_{p^r}\} \subset G \times X$

↑ an element of E

$\sigma \in G$

$$\sigma.\left(\{e_1, \dots, e_{p^r}\}\right) = \{\sigma(e_1), \sigma(e_2), \dots, \sigma(e_{p^r})\}$$

another elt. of E

Now E = disjoint union of orbits. For $\mathcal{O} \in G \backslash E$

$|\mathcal{O}|$ divides $|G| = p^r \Rightarrow |\mathcal{O}| = 1$
or p divides $|\mathcal{O}|$.

So $\boxed{|E| = \# \text{ of orbits with exactly one elt} \pmod p}$.

= m by

Ex. Orbits in E with exactly one element

$$= \left\{ \{(g, x_1) : g \in G\} ; \{(g, x_2) : g \in G\} ; \dots ; \{(g, x_m) : g \in G\} \right\}$$

□