

Lecture 11

(11.0) Recall some counting tricks we learnt :

$$G \curvearrowright X \quad \rightsquigarrow \quad \boxed{\begin{array}{ccc} G/\text{Stab}_G(x) & \xrightarrow{\text{bijection}} & G \cdot x \\ g \cdot \text{Stab}(x) & \longmapsto & g \cdot x \end{array}} \quad \textcircled{\text{I}}$$

$$\textcircled{\text{II}} \quad \boxed{\begin{array}{ccc} \text{Stab}(x) & \xrightarrow{\text{gp. iso.}} & \text{Stab}(\sigma x) \\ \psi \downarrow & & \downarrow \psi \\ g & \longmapsto & \sigma g \sigma^{-1} (= \text{Conj}(\sigma)(g)) \end{array}}$$

$$\left[\text{also } \begin{array}{ccc} X^g & \xrightarrow{\text{bijection}} & X^{\sigma g \sigma^{-1}} \\ \psi \downarrow & & \downarrow \psi \\ x & \longmapsto & \sigma x \end{array} \quad \forall \sigma, g \in G. \right]$$

Lemma (10.3) page 3 :

$$\boxed{\frac{|X|}{|G|} = \sum_{\mathcal{O} \in G \backslash X} \frac{1}{|\text{Stab}_G(x)|}} \quad \textcircled{\text{III}}$$

↑ set of orbits

an elt. from \mathcal{O}

See Example (10.7) page 6.

(11.1) Burnside's Theorem (Section (10.4) page 4).

$$|G \backslash X| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

One proof of Burnside's theorem is given on page 4 of Lecture 10. Here is another one - using a bit of linear algebra. [Ignore if you are not comfortable with it.]

Idea: change $|X|$ to dimension of a (real) vector space
= functions $\{X \rightarrow \mathbb{R}\}$

Functions $(G \setminus X \rightarrow \mathbb{R})$

= Functions $\{f: X \rightarrow \mathbb{R} \text{ such that } f(g \cdot x) = f(x) \forall g \in G, x \in X\}$

\subset all functions $f: X \rightarrow \mathbb{R}$

concretely, a vector space with basis $\{e_x : x \in X\}$ (so of $\dim = |X|$) and G -action

$\rho: G \rightarrow GL_{N(=|X|)}(\mathbb{R}); \rho(g): e_x \mapsto e_{g \cdot x}$
Ex. Trace of $\rho(g) = |X^g|$
[$(x, y)^{th}$ matrix entry of $\rho(g)$ is 1 iff $x = gy$; 0 otherwise]

So, the identity we have to prove is

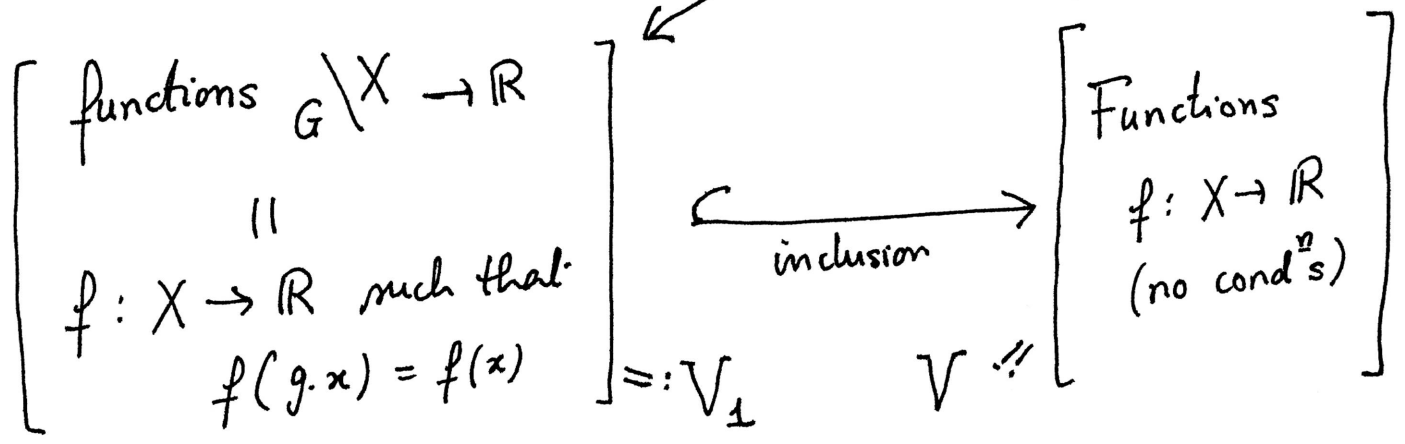
dimension (functions $G \setminus X \rightarrow \mathbb{R}$)

= Trace of $\left(\frac{1}{|G|} \sum_{g \in G} \rho(g) \right)$

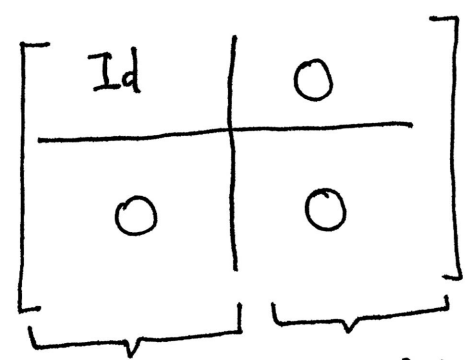
Averaging operator.

Now we have

$$Av = \frac{1}{|G|} \sum_{g \in G} \rho(g)$$



The matrix $\text{Inclusion} \circ Av : V \rightarrow V$ has the form

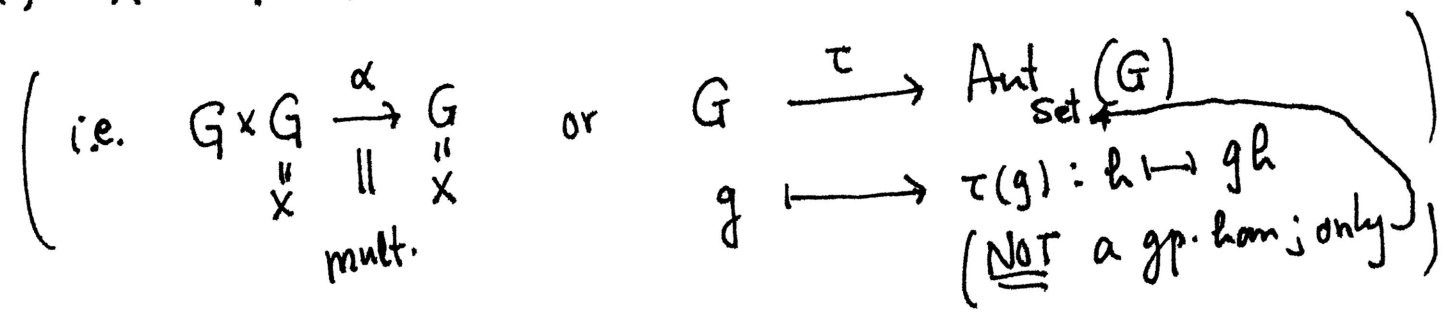


a basis of V_1 basis of $\text{Kernel}(Av) = \{v \in V \mid Av(v) = 0\}$

So its trace = size of top-left block = dimension of V_1 □

(11.2) Some "abstract" examples of $G \curvearrowright X$.

(i) $X = G$, G acts as multiplication on the left



Slightly more generally $G \curvearrowright G/H = \text{set of left cosets}$

(4)

by $g \cdot (g'H) := (gg')H$
 acting on \uparrow
 a typical elt. of G/H

$$\text{Stab}_G(gH) = \left\{ \sigma \in G \text{ st. } \begin{array}{l} \sigma gH = gH \\ \text{i.e. } g^{-1}\sigma g \in H \\ \text{i.e. } \sigma \in gHg^{-1} \end{array} \right\}$$

Let us make a note of this:

$$\begin{array}{l} \sigma \cdot (gH) = gH \\ \Leftrightarrow \sigma \in gHg^{-1} \end{array}$$

$G \curvearrowright G/H$ is transitive (so only one orbit = G/H)

$$\forall \sigma \in G, (G/H)^{\sigma} = \{ gH : \sigma \in gHg^{-1} \}$$

(fixed pts.)

(ii) $X = G$, G acts by conjugation

$$\sigma \cdot g = \text{Conj}(\sigma)(g) = \sigma g \sigma^{-1}$$

[Recall: $\text{Conj} : G \longrightarrow \underset{\text{Group}}{\text{Aut}}(G) \subset \underset{\text{set}}{\text{Aut}}(G)$]

Orbits of $G \curvearrowright G$ via conjugation are called conjugacy classes (5)

[A word on notation. - since $X = G$, we may get confused.
 I will try to keep g, g_1, g_2 - for elements of G - viewed as the group
 x, x_1, x_2 - for elements of G - viewed as a set on which the group is acting]

$x \in G (= X)$. $\text{Stab}(x) = \{ g \in G \mid gxg^{-1} = x \}$
 ↑
 called centralizer of x ,
 denoted by $Z_G(x)$.

$g \in G$. [so, $G / Z(x) \xrightarrow{\text{bij}}$ Conjugacy class of x]

Fixed pts of g under conjugation
 $\neq // \text{Ker}(\text{Conj}(g)) \neq \# \rightarrow \text{also} = Z(g)$

Kernel of $\text{Conj} : G \rightarrow \text{Aut}_{\text{Group}}(G) \subset \text{Aut}_{\text{Set}}(X=G)$
 $= \{ g \in G \mid gx = xg \ \forall x \in G \}$
 $= Z(G) = \underline{\text{center of } G}$.

Counting results for $G \curvearrowright G$ by conjugation: ⑥

(1) (see eqⁿ III of page 1)

$$1 = \sum_{C: \text{conjugacy class}} \frac{1}{|Z_G(\sigma_c)|}$$

(0) $|C| = \frac{|G|}{|Z_G(\sigma_c)|}$

$\sigma_c \in C$
a choice

(2) Burnside's Thm (see the statement on page 1)

$$\# \text{ of Conjugacy class} = \frac{1}{|G|} \sum_{g \in G} |Z_G(g)|$$

(sometimes people call this "Class Equation".)

(11.3) Fun identities.

Take $G = S_n \curvearrowright X = S_n$ by conjugation

Q. What are conjugacy classes?

A. Recall: $\sigma \cdot (x_1 x_2 \dots x_\ell) \sigma^{-1} = (\sigma(x_1) \sigma(x_2) \dots \sigma(x_\ell))$

So Conjugacy classes in $S_n \iff$ Cycle types (i.e. "partitions of n ")

Usually people label conjugacy classes as $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r)$
($\lambda_1 + \lambda_2 + \dots + \lambda_r = n$)
Cycle types organised in non-decreasing order

Or $\lambda = (\underbrace{1, 1, \dots, 1}_{l_1 \text{ times}} ; \underbrace{2, 2, \dots, 2}_{l_2 \text{ times}} ; \dots)$ ⑦

abbreviated as $(1^{l_1}, 2^{l_2}, 3^{l_3}, \dots)$

(so $(1^{l_1}, 2^{l_2}, 3^{l_3}, \dots, n^{l_n})$ is a valid cycle type for a permutation $\sigma \in S_n$

$\Leftrightarrow 1 \cdot l_1 + 2 \cdot l_2 + 3 \cdot l_3 + \dots + n \cdot l_n = n$)

Q.. How many elements ^{in S_n} are there of given cycle type?

A. Cycle type $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r)$
 or $(1^{l_1}, 2^{l_2}, 3^{l_3}, \dots)$

$C_\lambda = \{ \sigma \in S_n \mid \sigma \text{ is of cycle type } \lambda \}$

$|C_\lambda| = \frac{n!}{\lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_r \cdot l_1! \cdot l_2! \cdot \dots \cdot l_n!}$

$|C_\lambda| = \frac{n!}{1^{l_1} \cdot 2^{l_2} \cdot 3^{l_3} \cdot \dots \cdot (l_1! \cdot l_2! \cdot \dots)}$