Lecture 11

(11.0) Recall some counting tricks we learnt :

$$G \circlearrowright X \quad \rightsquigarrow \quad \boxed{\begin{array}{ccc} G/\mathrm{Stab}_G(x) & \xrightarrow{\text{bijection}} & G.x \\ g.\mathrm{Stab}(x) & \longmapsto & g.x \end{array}} \quad \text{I}$$

$$\& \quad \boxed{\begin{array}{ccc} \mathrm{Stab}(x) & \xrightarrow[\sim]{\text{gp. iso.}} & \mathrm{Stab}(\sigma x) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ g & \longmapsto & \sigma g \sigma^{-1} \ (= \mathrm{Conj}(\sigma)(g)) \end{array}} \quad \text{II}$$

$$\left[\text{also} \quad \begin{array}{ccc} X^{g} & \xrightarrow{\text{bijection}} & X^{\sigma g \sigma^{-1}} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longrightarrow & \sigma x \end{array} \quad \forall\, \sigma, g \in G. \right]$$

Lemma (10.3) page 3 :

$$\boxed{\frac{|X|}{|G|} = \sum_{\mathcal{O} \in G \backslash X} \frac{1}{|\mathrm{Stab}_G(x)|}} \quad \text{— III}$$

an elt. from $\mathcal{O}$

set of orbits

See Example (10.7) page 6.

(11.1) Burnside's Theorem (Section (10.4) page 4).

$$|G \backslash X| = \frac{1}{|G|} \sum_{g \in G} |X^{g}|$$

One proof of Burnside's theorem is given on page 4 of Lecture 10. Here is another one - using a bit of linear algebra. [Ignore if you are not comfortable with it.]

Idea : change $|X|$ to dimension of a (real) vector space

$= \text{functions } \{X \to \mathbb{R}\}$

$\text{Functions}\left(G\backslash X, \mathbb{R}\right)$

$= \text{Functions } \{ f : X \longrightarrow \mathbb{R}$ such that $f(g.x) = f(x)$ $\forall\ g \in G,\ x \in X \}$

$\subset$ all functions $f : X \to \mathbb{R}$

concretely, a vector space with basis $\{e_x : x \in X\}$ (so of dim $= |X|$) and G- action

$\rho : G \to GL_{N(=|X|)}(\mathbb{R})$; $\rho(g) : e_x \longmapsto e_{g.x}$ [$(x,y)^{th}$ matrix entry of $\rho(g)$ is 1 iff $x = gy$; 0 otherwise]

Ex. Trace of $\rho(g) = |X^g|$

So, the identity we have to prove is

$$\text{dimension}\left(\text{functions } G\backslash X \to \mathbb{R}\right)$$

$$= \text{Trace of}\left(\frac{1}{|G|}\sum_{g \in G} \rho(g)\right)$$

Averaging operator.

Now we have

$$\begin{tikzcd}[column sep=huge]
\left[\begin{array}{c}\text{functions } {}_G\backslash X \to \mathbb{R} \\ \| \\ f : X \to \mathbb{R} \text{ such that} \\ f(g.x) = f(x)\end{array}\right] =: V_1 \arrow[r, hook, "\text{inclusion}"'] & V =: \left[\begin{array}{c}\text{Functions} \\ f : X \to \mathbb{R} \\ (\text{no cond}^{\underline{n}}\text{s})\end{array}\right] \arrow[l, bend right, "Av = \frac{1}{|G|}\sum_{g\in G}\rho(g)"']
\end{tikzcd}$$

The matrix $\text{Inclusion} \circ Av : V \longrightarrow V$ has the form

$$\left[\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$\underbrace{\qquad\qquad}_{\text{a basis of } V_1} \quad \underbrace{\qquad\qquad}_{\text{basis of Kernel}(Av) = \{v \in V \mid Av(v) = 0\}}$$

So its trace $=$ size of top. left block $=$ dimension of V_1

$\square$

(11.2) Some "abstract" examples of $G \circlearrowright X$.

(i) $X = G$, G acts as multiplication on the left

$$\left(\text{i.e. } \begin{array}{ccc} G \times G & \xrightarrow{\alpha} & G \\ \| & \| & \| \\ X & \text{mult.} & X \end{array} \quad \text{or} \quad \begin{array}{rcl} G & \xrightarrow{\tau} & \mathrm{Aut}_{set}(G) \\ g & \longmapsto & \tau(g) : h \mapsto gh \end{array} \right)$$

($\underline{\underline{\text{NOT}}}$ a gp. hom; only)

Slightly more generally $G \circlearrowright G/H$ = set of left cosets

by $g \cdot (g' \cdot H) := (gg')H$.

acting on

a typical elt. of G/H

$$\operatorname{Stab}_G(gH) = \left\{ \sigma \in G \text{ s.t. } \begin{array}{l} \sigma g H = gH \\ \text{i.e. } g^{-1}\sigma g \in H \\ \text{i.e. } \sigma \in gHg^{-1} \end{array} \right\}$$

Let us make a note of this: $\boxed{\begin{array}{c} \sigma \cdot (gH) = gH \\ \iff \sigma \in gHg^{-1} \end{array}}$

$G \circlearrowright G/H$ is transitive (so only one orbit $= G/H$)

$\forall \sigma \in G, \quad (G/H)^{\sigma} = \{ gH : \sigma \in gHg^{-1} \}$

(fixed pts.)

(ii) $X = G$, G acts by conjugation

$$\sigma \bullet g = \operatorname{Conj}(\sigma)(g) = \sigma g \sigma^{-1}$$

[Recall: $\operatorname{Conj} : G \longrightarrow \operatorname{Aut}_{\text{Group}}(G) \subset \operatorname{Aut}_{\text{set}}(G)$.]

Orbits of $G \circlearrowright G$ via conjugation are called <u>conjugacy classes</u>

[A word on notation. – since $X = G$, we may get confused.
I will try to keep g, g_1, g_2 – for elements of G – viewed as <u>the group</u>

x, x_1, x_2 – for elements of G – viewed as a <u>set</u> on which the group is acting]

$x \in G \ (= X)$. $\quad \mathrm{Stab}(x) = \{ g \in G \mid g x g^{-1} = x \}$

called <u>centralizer of x</u>, denoted by $Z_G(x)$.

$g \in G$. $\quad$ [so, $G / Z(x) \xrightarrow{\text{bij}}$ Conjugacy class of x]

Fixed pts of g under conjugation

$\cancel{\# \, |\mathrm{Ker}(\mathrm{Conj}(g))|} \quad \cancel{\#}$ also $= Z(g)$.

Kernel of $\mathrm{Conj} : G \longrightarrow \mathrm{Aut}_{\text{Group}}(G) \subset \mathrm{Aut}_{\text{Set}}(X = G)$

$= \{ g \in G \mid g x = x g \ \ \forall x \in G \}$

$= Z(G) =$ <u>center of G</u>.

Counting results for $G \circlearrowright G$ by conjugation:

(1) (see eq^n III of page 1)

$$1 = \sum_{C:\text{ conjugacy class}} \frac{1}{|Z_G(\sigma_C)|}$$

$\sigma_C \in C$ a choice.

(0) $\boxed{|C| = \dfrac{|G|}{|Z_G(\sigma_C)|}}$

(2) Burnside's Thm (see the statement on page 1)

$$\#\text{ of Conjugacy class} = \frac{1}{|G|} \sum_{g \in G} |Z_G(g)|$$

(sometimes people call this "Class Equation".)

(11.3) Fun identities.

Take $G = S_n \circlearrowright X = S_n$ by conjugation

Q. What are conjugacy classes?

A. Recall: $\sigma \cdot (x_1\, x_2 \ldots x_\ell)\, \sigma^{-1} = (\sigma(x_1)\; \sigma(x_2) \ldots \sigma(x_\ell))$

So Conjugacy classes in S_n $\longleftrightarrow$ Cycle types (i.e. "partitions of n")

Usually people label conjugacy classes as either $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r)$ $(\lambda_1 + \lambda_2 + \ldots + \lambda_r = n)$

Cycle types organised in non-decreasing order

Or $\lambda = (\underbrace{1,1,\dots,1}_{l_1 \text{ times}} ; \underbrace{2,2,\dots,2}_{l_2 \text{ times}} ; \dots)$

abbreviated as $(1^{l_1}, 2^{l_2}, 3^{l_3}, \dots)$

(so $(1^{l_1}, 2^{l_1}, 3^{l_3}, \dots, n^{l_n})$ is a valid cycle type for a permutation $\sigma \in S_n$

$$\iff 1 \cdot l_1 + 2 \cdot l_2 + 3 \cdot l_3 + \dots + n \cdot l_n = n \quad)$$

Q.. How many elements in S_n are there of given cycle type?

A. Cycle type $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r)$ or $(1^{l_1}, 2^{l_2}, 3^{l_3} \dots)$

$$C_\lambda = \{ \sigma \in S_n \mid \sigma \text{ is of cycle type } \lambda \}$$

$$|C_\lambda| = \frac{n!}{\lambda_1 \cdot \lambda_2 \cdots \lambda_r \cdot l_1! \, l_2! \cdots l_n!}$$

$$\boxed{|C_\lambda| = \frac{n!}{1^{l_1} \cdot 2^{l_2} \cdot 3^{l_3} \dots (l_1! \, l_2! \cdots)}}$$