Lecture 12
(most of it is optional.)

(12.0) Recall: (i) Conjugacy classes in S_n ← symmetric gp. on n letters

$\longleftrightarrow$ Cycle types

(recall :- $\sigma, \tau \in S_n$ ~~hdabt~~ are in the same conjugacy class

$\iff \sigma = w \tau w^{-1}$ for some $w \in S_n$.)

Cycle types $\lambda = (\lambda_1 \ \lambda_2 \ \lambda_3 \ \dots \ \lambda_r)$ $\longrightarrow$ arranged from larger to smaller $\left(\begin{array}{l} l_i = \# \text{ of } \lambda_j\text{'s which are equal to } i \\ 1 \le i \le n \end{array} \right)$

where $\lambda_1 + \lambda_2 + \dots + \lambda_r = n$

$C_\lambda \overset{\text{defn}}{:=} (C(\lambda) =) \#$ of permutation in S_n of cycle type λ.

$$= \frac{n!}{Z_\lambda} \quad \text{where } Z_\lambda = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_r \cdot l_1! \, l_2! \cdots l_n!$$
$$= 1^{l_1} \cdot 2^{l_2} \cdots n^{l_n} \cdot l_1! \, l_2! \cdots l_n!$$

↑ proved in last class (see (12.1) below)

We noticed two identities

$$\sum_{\lambda : \text{cycle type}} \frac{1}{Z_\lambda} = 1 = \sum_{\lambda : \text{cycle type}} \frac{l_1}{Z_\lambda}$$

using ($G = S_n \overset{\text{conj.}}{\circlearrowright} X = S_n$)

$$|X| = \sum_{\mathcal{O} : \text{orbit}} |\mathcal{O}| = |G| \cdot \sum_{\mathcal{O} : \text{orbit}} \frac{1}{|\mathrm{stab}_G(x_{\mathcal{O}})|}$$

← an elt. from $\mathcal{O}$.

using Burnside's Thm (for $G = S_n \circlearrowright X = \{1, \dots, n\}$)

(12.1) Proof. of $c_\lambda = \frac{n!}{Z_\lambda}$; $Z_\lambda = 1^{\ell_1} \cdot 2^{\ell_2} \cdots n^{\ell_n} \cdot \ell_1! \cdots \ell_n!$

$S_n \circlearrowright S_n$ Conjugation, $\sigma \in S_n$, $x \in S_n$

$\sigma \bullet x = \sigma x \sigma^{-1}$

Ex. Orbit of any ~~partition~~ permutation of type λ.
= all permutations of type λ.
denoted henceforth by $\mathcal{O}_\lambda$:

$\boxed{\mathcal{O}_\lambda \xleftarrow{\text{bijection}} S_n / \text{centralizer of } \pi_\lambda} \Rightarrow c_\lambda \cdot Z_\lambda = n!$

Our argument : to build a permutation of type λ –

order $\{1, \ldots, n\}$ arbitrarily ($\underline{\underline{n! \text{ choices}}}$)

$\rightsquigarrow (x_1 \; x_2 \; \cdots \; x_{\lambda_1})(x_{\lambda_1+1} \cdots x_{\lambda_1+\lambda_2}) \cdots (x_{\lambda_1+\cdots+\lambda_{r-1}+1} \cdots x_{\lambda_1+\cdots+\lambda_r})$, $\lambda_1+\cdots+\lambda_r = n$

Overcounting 1 : $(x_1 \; x_2 \; \ldots \; x_\ell) = (x_i \; x_{i+1} \cdots x_\ell \; x_1 \cdots x_{i-1})$

So $\underline{\underline{\text{divide by } \lambda_1 \cdot \lambda_2 \cdots \lambda_r}}$

$= 1^{\ell_1} \cdot 2^{\ell_2} \cdots n^{\ell_n}$
by defn. of $\ell_1, \ell_2, \ldots, \ell_n$
$(\ell_i = \#\{j \mid \lambda_j = i\})$

Overcounting 2:
cycles of same length can be permuted

So divide by $\ell_1! \cdots \ell_n!$

$$\Rightarrow \quad c_\lambda = \frac{n!}{1^{\ell_1} 2^{\ell_2} \cdots n^{\ell_n} \; \ell_1! \cdots \ell_n!} \qquad \square$$

(12.2) A cool proof of $\displaystyle\sum_{\substack{\lambda:\text{ cycle type}\\(\text{in } S_n)}} \frac{1}{z_\lambda} = 1$

Idea: form a sum $\displaystyle f(x) = 1 + \sum_{n\geq 1}\Bigg(\sum_{\substack{\lambda:\text{ cycle}\\ \text{type in}\\ S_n}} \frac{1}{z_\lambda}\Bigg) x^n$

$$f(x) = 1 + \sum_{n\geq 1}\Bigg(\sum_{\substack{l_1, l_2, \ldots, l_n \geq 0\\ \text{such that}\\ 1\cdot l_1 + 2\cdot l_2 + \ldots + n\cdot l_n = n}} \frac{1}{1^{l_1}\cdot 2^{l_2}\cdots n^{l_n}\cdot l_1!\, l_2! \cdots l_n!}\Bigg) x^n$$

defn. of z_λ

$$= 1 + \sum_{n\geq 1}\Bigg(\sum_{\substack{l_1, l_2, \ldots, l_n \geq 0\\ \text{such that}\\ 1\cdot l_1 + 2\cdot l_2 + \ldots + n\cdot l_n = n}} \frac{(x^1)^{l_1}(x^2)^{l_2}(x^3)^{l_3}\cdots (x^n)^{l_n}}{1^{l_1}\cdot l_1!\; 2^{l_2}\cdot l_2!\; 3^{l_3} l_3! \cdots n^{l_n}\cdot l_n!}\Bigg)$$

because $n = 1\cdot l_1 + 2\cdot l_2 + \ldots + n\, l_n$

$$= \sum_{l_1, l_2, l_3, \ldots \geq 0} \left(\frac{1}{l_1!}\left(\frac{x^1}{1}\right)^{l_1}\right)\left(\frac{1}{l_2!}\left(\frac{x^2}{2}\right)^{l_2}\right)\cdots\left(\frac{1}{l_k!}\left(\frac{x^k}{k}\right)^{l_k}\right)\cdots$$

such that almost all are zero

meaning $1\cdot l_1 + 2\cdot l_2 + 3\cdot l_3 + \ldots$ must be finite $\longleftrightarrow$ $\{j \mid l_j \neq 0\}$ is finite

Now keep $l_2, l_3, \ldots$ fixed and first sum over l_1

$$f(x) = \sum_{\substack{l_2, l_3, \ldots \geq 0\\ \text{almost all are zero}}} \left\{\sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{x^1}{1}\right)^{l}\right\} \frac{1}{l_2!}\left(\frac{x^2}{2}\right)^{l_2} \cdots \frac{1}{l_k!}\left(\frac{x^k}{k}\right)^{l_k} \cdots$$

exponential $e^{x^1/1}$ $(= e^x)$

keep doing the same one sum at a time to get

$$f(x) = e^{x} \cdot e^{x^2/2} \cdot e^{x^3/3} \cdots$$

$$= e^{x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots}$$ Taylor series of $-\log(1-x)$

$$= e^{-\log(1-x)} \qquad \left(-\log(t) = \log(t^{-1})\right)$$

$$= e^{\log((1-x)^{-1})} = (1-x)^{-1} \qquad \left(e^{\log(t)} = t\right)$$

$$= \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Meaning :- Coefficient of x^n in $f(x) = 1$

but we defined it to be $\displaystyle\sum_{\substack{\lambda:\text{ cycle type}\\ \text{in } S_n}} \frac{1}{Z_\lambda}$. $\square$

(12.3) What to change to get $\displaystyle\sum_{\substack{\lambda:\text{ cycle type}\\ \text{in } S_n}} \frac{l_1}{Z_\lambda} = 1$?

① Define $g(x) = \displaystyle\sum_{n \geq 1} \left(\sum_{\substack{\lambda:\text{ cycle type}\\ \text{in } S_n}} \frac{l_1}{Z_\lambda} \right) x^n$ as before.

2 Carry out the same steps for $f(x) = \dots$ on page 3, for $g(x)$. You should get

this is new.

$$g(x) = \sum_{\substack{l_1, l_2, \dots \geq 0 \\ \text{atmost all zero}}} \left\{ \frac{l_1}{l_1!} \left(\frac{x^1}{1}\right)^{l_1} \right\} \underbrace{\left(\frac{1}{l_2!} \left(\frac{x^2}{2}\right)^{l_2} \right) \left(\frac{1}{l_3!} \left(\frac{x^3}{3}\right)^{l_3} \right) \cdots}_{\text{rest same as before.}}$$

$$= \underbrace{\left(x^1 \cdot e^{x^1/1} \right)}_{\text{new}} \underbrace{\left(e^{x^2/2} \right) \left(e^{x^3/3} \right) \cdots}_{\text{rest same as before}}$$

$$= \quad \dots \quad = \frac{x}{1-x} = x + x^2 + x^3 + \dots$$

(12.4) How far can we take this?

We can change $\frac{x^k}{k}$ in the second line of the eq^n simplifying $f(x)$ on page 3, by a new variable p_k and same calculation carries through.

product of variables : $p_\lambda = p_1^{l_1} p_2^{l_2} p_3^{l_3} \cdots$

$$\sum_{\substack{l_1, l_2, l_3, \dots \geq 0 \\ \text{almost all zero}}} \frac{1}{z_\lambda} \cdot p_\lambda = \prod_{r \geq 1} e^{p_r/r} = e^{p_1 + \frac{p_2}{2} + \frac{p_3}{3} + \dots}$$

or λ : cycle type

One way to use this – e.g. – to answer Ishaan's question –

$$\boxed{\sum_{\substack{\lambda:\text{ cycle type}\\ \text{in } S_n}} \frac{\ell_k}{z_\lambda}} = ?$$

$$\text{Coefficient of } x^{n-k} \text{ in } \left[\frac{\partial}{\partial p_k} \left\{ \sum_{\lambda} \frac{p_\lambda}{z_\lambda} \right\} \right]_{\text{set } p_j \overset{\text{back}}{=} x^j \ \forall j}$$

$$= \text{Coefficient of } x^{n-k} \text{ in } \left[\frac{\partial}{\partial p_k} e^{p_1 + \frac{p_2}{2} + \dots + \frac{p_k}{k} + \dots} \right]_{\text{set } p_j = x^j \ \forall j}$$

$$= \quad '' \quad \text{in } \left[e^{p_1 + \frac{p_2}{2} + \dots} \cdot \frac{1}{k} \right]_{\text{set } p_j = x^j \ \forall j}$$

$$= \frac{1}{k} \cdot \left\{ \text{coefficient of } x^{n-k} \text{ in } e^{x + \frac{x^2}{2} + \frac{x^3}{3} + \dots} = \frac{1}{1-x} = 1 + x + x^2 + \dots \right\}$$

$$= \frac{1}{k} \quad \text{iff } k \le n$$