Lecture 17

(17.0) Sylow subgroups in D_{2m} (m = an odd number).

$$D_{2m} = \left\{ \boxed{e,\ r, r^2, \ldots, r^{m-1}} \quad \boxed{s, sr, sr^2, \ldots, sr^{m-1}} \right\}$$

elements of order 2.

normal subgroup of D_{2m}

$$\langle r \rangle \cong \mathbb{Z}/m\mathbb{Z}$$

$$Syl_2(D_{2\cdot m}) = \left\{ \{e, s.r^j\} : 0 \leq j \leq m-1 \right\}$$

$$\Rightarrow \quad n_2(D_{2m}) = m$$

no order 2 element can be in Q, as $|Q|$ is odd

Let p be an odd prime (dividing m). If $Q \in Syl_p(D_{2m})$, then

$$\boxed{Q \subseteq \underset{\substack{\uparrow \\ \text{abelian.}}}{\langle r \rangle}} \trianglelefteq D_{2\cdot m}$$

$\hookrightarrow Q \in Syl_p(\mathbb{Z}/m\mathbb{Z})$

Claim : $n_p(D_{2m}) = 1$.

[Reason: $Syl_p(\mathbb{Z}/m\mathbb{Z}) = 1$ because it is abelian.
If $Q_1, Q_2 \in Syl_p(D_{2m})$ then they are both Sylow p-subgroups of $\mathbb{Z}/m\mathbb{Z}$.]

(17.1) Sylow 2-subgroups in $D_{2^a \cdot m}$ $\left(\begin{matrix} a \geq 1 \\ m : \text{odd} \end{matrix}\right)$.

e.g. $D_{2\cdot(2\cdot 3)} = \langle s, r \mid s^2 = r^6 = e \,;\, srs = r^{-1} \rangle$

Ψ

r^3 is <u>order 2</u> <u>central element</u>. It must be in

<u>all</u> Sylow 2-subgroups:

$\{e, s, r^3, sr^3\}$; $\{e, sr, r^3, sr^4\}$; $\{e, sr^2, r^3, sr^5\}$

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$\{e, sr^3, r^3, sr^6\}$; $\{e, sr^4, r^3, \underset{\overset{\|}{sr}}{sr^7}\}$; $\{e, sr^5, r^3, \underset{\overset{\|}{sr^2}}{sr^8}\}$

i.e. "mod out by $\underset{\overset{\|}{Z(D_{2\cdot 6})}}{\langle r^3 \rangle} \cong \mathbb{Z}/2$" and then count.

Answer: $\quad n_2\left(D_{2^a\cdot m}\right) = m.$

Proof. $\quad D_{2^{a+1}\cdot m} \xrightarrow{\quad f \quad} D_{2^a\cdot m} \cong D_{2^{a+1}\cdot m} \Big/ \langle r^{2^{a-1}\cdot m} \rangle$

$\uparrow$ center of $D_{2^{a+1}\cdot m}$

$|\mathrm{Ker}(f)| = 2.$

Ex. f gives a bijection between

$$\mathrm{Syl}_2\left(D_{2^{a+1}\cdot m}\right) \longleftrightarrow \mathrm{Syl}_2\left(D_{2^a\cdot m}\right) \qquad \square$$

$$\cdots \longleftrightarrow \mathrm{Syl}_2\left(D_{2m}\right) = m \text{ by } (17.0)$$

(17.2) $G = GL_n(\mathbb{F}_p)$. $n \geq 2$; p: prime.

$\left[\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \right.$ $\longleftarrow$ usual $+$ and $\cdot$ of $\mathbb{Z}$ performed modulo p - a fixed prime $\left.\right]$

$$|G| = \underbrace{(p^n - 1)}_{\substack{\uparrow \\ \text{options for } 1^{st} \text{ column}}} \; (p^n - p) \; \cdots \; (p^n - p^{n-1})$$

$$= \boxed{p^{\frac{n(n-1)}{2}}} \cdot \left\{ (p^n - 1)(p^{n-1} - 1) \cdots (p-1) \right\}$$

$$U := \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in GL_n(\mathbb{F}_p) \right\}$$

$\#$ of spots $= \frac{n(n-1)}{2}$

so $|U| = p^{\frac{n(n-1)}{2}}$.

$U \in Syl_p(GL_n(\mathbb{F}_p))$.

$\#$ of Sylow p-subgroups in $GL_2(\mathbb{F}_p) = p + 1$.

$\left(p \cdot (p^2-1)(p-1) \text{ elements} \right)$

For every fixed $\alpha \in \mathbb{F}_p$:

$$\left\{ \begin{bmatrix} 1 + \alpha y & -\alpha^2 y \\ y & 1 - \alpha y \end{bmatrix} : y \in \mathbb{F}_p \right\}$$

$$\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}^{-1}$$

and $P_0 := \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{F}_p \right\}$. Total $p+1$ ✓.

How to think about this? Here is a cool fact you may have heard:

$\forall \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: 2×2 matrix s.t. $\Delta = ad - bc \neq 0$

either $c = 0$ or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & -\frac{\Delta}{c} \end{bmatrix}}_{\parallel}$$

$$\frac{ad - (ad - bc)}{c} = b \qquad \begin{bmatrix} a & \frac{ad}{c} - \frac{\Delta}{c} \\ c & d \end{bmatrix} = \begin{bmatrix} a/c & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} c & d \\ 0 & -\frac{\Delta}{c} \end{bmatrix}$$

Meaning $GL_2 = \boxed{\text{Upper triangular matrices}} \sqcup B \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot B$

$\parallel$

B (subgroup)

$$Syl_p(GL_2(\mathbb{F}_p)) = \left\{ g \cdot P_0 \cdot g^{-1} : g \in GL_2(\mathbb{F}_p) \right\}$$

$$g P_0 g^{-1} = P_0 \quad \forall g \in B$$

So let us conjugate P_0 by $g \in B \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} B$

$$g = A_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A_2 \qquad (A_1, A_2 : \text{upper triangular})$$

$$g P_0 g^{-1} = A_1 \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \underbrace{\boxed{A_2 P_0 A_2^{-1}}}_{\text{still } P_0} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\left\{ \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} : y \in \mathbb{F}_p \right\}} A_1^{-1}$$

So our only option is to conjugate $\left\{ \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} : y \in \mathbb{F}_p \right\}$ by an element of the form $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ (convince yourself that diagonal entries can be assumed to be 1).

(17.3) $G = S_p$ $\quad p$: prime

$$|G| = p! = \big((p-1)!\big) \cdot p^1$$

Sylow p-subgp. has size p.

$P \leq G$ Sylow p-subgroup $\Rightarrow |P| = p$

(i.e. $P \cong \mathbb{Z}/p\mathbb{Z}$)

P has $p-1$ elements of order p.

If $Syl_p(S_p) = \{P_1, \dots, P_{n_p}\}$, then

$P_i \cap P_j = \{e\}$ $(i \neq j)$ $\Rightarrow$ # of elements of order p in S_p

$= (p-1)\, n_p$

But # of elements in S_p of order p = # cycles of length p

$= (p-1)!$

$$\Rightarrow \boxed{n_p(S_p) = (p-2)!}$$