Lecture 18

(18.0) 3^{rd} Isomorphism Theorem.

Let G be a group, $H \leq G$ a subgroup and $N \trianglelefteq G$ a normal subgroup. Then

$$H/_{H\cap N} \xrightarrow[\text{group iso.}]{\cong} H.N \text{ (or } N.H)/_{N} .$$

Part of the statement of the theorem is :

- $H \cdot N = N \cdot H$ is a subgroup of G (containing N as a normal subgroup).

[Proof. − $n \cdot h = h \cdot \underbrace{\boxed{h^{-1}(n)\, h}}_{\in N}$ $\forall\, h \in H,\ n \in N$. since N is normal.]

- $H \cap N$ is normal in H

[Proof. − Let $h \in H$ and $x \in H \cap N$. Then

$h x h^{-1} \in H$ because H is a subgroup.
$\in N$ because N is a normal subgroup.]

Proof of the theorem. Consider the natural projection

$$\begin{array}{ccc} H & \longrightarrow & H\cdot N/_{N} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ h & \longmapsto & h\cdot N \end{array}$$

$H \xrightarrow{f} H.N/N$ is composed of $H \subset H.N$ (inclusion of a subgroup) and $H.N \longrightarrow H.N/N$ (as $N \subset H.N$ is normal.)

- $\mathrm{Ker}(f) = \{ h \in H$ such that $h.N = N \}$
  $= H \cap N.$
- $\mathrm{Im}(f) = H.N/N$ $\leftarrow$ every element here is of the form $h.N$ for some $h \in H$.

By 1^{st} iso. thm. (Source/Kernel = Image) we get an iso.

$$\begin{array}{ccc} H/H\cap N & \xrightarrow[\cong]{\bar{f}} & H.N/N \\ h.(H\cap N) & \longmapsto & h.N \end{array}$$

$\square$

(18.1) Definition. - We say that G is a <u>semi-direct product</u> of H and N if

(i) $H \leq G$ and $N \trianglelefteq G$.

(ii) $G = H.N = N.H$

(iii) $H \cap N = \{e\}$

Note. - If G is a semi-direct product of H and N, then by the theorem proved in (18.0) above,

$$\begin{array}{ccc} G/N & \xleftarrow{\cong} & H \\ \uplus & & \uplus \\ h\cdot N & \longleftarrow & h \end{array}$$

e.g. $G = D_{2n} \geq H = \{e, s\}$ a subgroup $\cong \mathbb{Z}/2\mathbb{Z}$

$\trianglerighteq N = \{e, r, r^2, \ldots, r^{n-1}\} \cong \mathbb{Z}/n\mathbb{Z}$

a normal subgroup

G = subgroup of $GL_2(\mathbb{C})$ consisting of upper triangular matrices.

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \text{ where } \begin{array}{l} a, c \in \mathbb{C} \setminus \{0\} \\ b \in \mathbb{C} \end{array} \right\}$$

$$G \geq H = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \text{ where } a_1, a_2 \in \mathbb{C} \setminus \{0\} \right\}$$

$$\trianglerighteq N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \text{ where } x \in \mathbb{C} \right\}$$

$G = H\cdot N = N\cdot H$ because

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & b/c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$$

(18·2) Where do semi-direct products come from?

Let G be a group with $21 = 3\cdot 7$ elements $\left(\begin{matrix} p < q \\ q \equiv 1 \bmod p \end{matrix}\right)$

$$\left.\begin{array}{l} n_3 \equiv 1 \ (3) \\ n_3 \mid 7 \end{array} \Rightarrow n_3 = 1 \text{ or } 7 \right]$$

Sylow Thm. (part 3)

$$\left.\begin{array}{l} n_7 \equiv 1 \ (7) \\ n_7 \mid 3 \end{array} \Rightarrow n_7 = 1 \right]$$

<u>Situation 1</u> : $n_3 = 1$, $n_7 = 1$ $\rightsquigarrow$ direct product

Meaning, since $Syl_3(G) = \{P\}$; $Syl_7(G) = \{Q\}$, both normal in G as $n_3 = n_7 = 1$.

we get (i) $G \trianglerighteq P, Q$ (ii) $P \cap Q = \{e\}$

(iii) $G = P\cdot Q$

Put together $G \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$ $(\cong \mathbb{Z}/21\mathbb{Z})$

<u>Situation 2</u> : $n_3 = 7$; $n_7 = 1$.

Let $P \in Syl_3(G)$ and $\{Q\} = Syl_7(G)$. still normal.

We still get $P \cap Q = \{e\}$ and $G = P\cdot Q$

but P need not be normal in G. ⇝ semidirect product.

Let us write $P = \{1, x, x^2\} \leq G$

$Q = \{1, y, y^2, \ldots, y^6\} \trianglelefteq G$

We obtain an isomorphism

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & Q \\ \Downarrow & & \Downarrow \\ q & \longmapsto & x q x^{-1} = \mathrm{Conj}(x)(q) \end{array}$$

about which we know the following properties :-

(i) $\varphi \circ \varphi \circ \varphi$ = Identity map $Q \longrightarrow Q$ (since $x^3 = e$)

(ii) φ is non-trivial, since $\varphi(y) = y$ means P & Q commute and that is Situation 1 above.

(iii) φ is an iso. of $Q \cong \mathbb{Z}/7\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/7\mathbb{Z} \cong Q$

$$\left\{ \begin{array}{ccc} \text{Group Iso's } \mathbb{Z}/7\mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z}/7\mathbb{Z} \\ & \Downarrow & \\ 1 & \longmapsto & t \in \{1,2,3,4,5,6\} \end{array} \right\} \leftarrow 6 \text{ elements}$$

⇝ To get order 3 (so as (i) is true) we need

$t = 1$ or $\underbrace{2 \text{ or } 4}_{}$ ← 2 possibilities

not an option, by (ii)