

Lecture 19

①

(19.0) Constructing semidirect products.

Assume we are given two groups H and N ; and a group homomorphism $\alpha: H \longrightarrow \text{Aut}_{\text{group}}(N)$

↑
set of all group iso $N \xrightarrow{\sim} N$

[Meaning: for every $h \in H$, $\alpha(h): N \longrightarrow N$ is an isomorphism of N with itself. ($\alpha(h)^{-1} = \alpha(h^{-1})$ see below.)

$$\alpha(e_H) = \text{Identity } \begin{array}{c} N \rightarrow N \\ \cup \\ n \mapsto n \end{array} \quad \text{and} \quad \alpha(h_1)(\alpha(h_2)(n)) = \alpha(h_1 h_2)(n).]$$

↑
unit of H

We can define a binary operation (2 inputs; 1 output)

on $G = \{ (n, h) \text{ where } n \in N \text{ and } h \in H \}$ as follows:

$$\boxed{(n_1, h_1) \cdot (n_2, h_2) = (n_1 \cdot \alpha(h_1)(n_2), h_1 h_2)} \quad \forall \begin{array}{l} n_1, n_2 \in N \\ h_1, h_2 \in H. \end{array}$$

Lemma. - The operation written above defines a group structure on the set $G = N \times H$.

Let us denote this group by $N \rtimes_{\alpha} H$.

Proof We need to check that (i) the operation defined above is associative; (ii) there is a neutral element and (iii) every element has an inverse.

(i). $g_1 = (n_1, h_1)$ $g_2 = (n_2, h_2)$ $g_3 = (n_3, h_3)$
 (three typical elements of $G = N \times H$ - cartesian product of sets)

$$(g_1 \cdot g_2) \cdot g_3 = (n_1 \cdot [\alpha(h_1)(n_2)] \cdot [\alpha(h_1 h_2)(n_3)], h_1 h_2 h_3)$$

$$g_1 \cdot (g_2 \cdot g_3) = (n_1 \cdot [\alpha(h_1)(n_2 \cdot [\alpha(h_2)(n_3)])], h_1 h_2 h_3)$$

$$\alpha(h_1)(n_2 \cdot \alpha(h_2)(n_3)) \stackrel{!}{=} \alpha(h_1)(n_2) \cdot \alpha(h_1)(\alpha(h_2)(n_3))$$

$\alpha(h_1)$ is a gp. hom.

$$= \alpha(h_1)(n_2) \cdot \alpha(h_1 h_2)(n_3) \quad \checkmark$$

$\alpha: H \rightarrow \text{Aut}_{gp} N$ is a gp. hom

$$\Rightarrow (g_1 g_2) g_3 = g_1 (g_2 g_3)$$

(ii) Let $e_1 \in H$, $e_2 \in N$ be the respective identity elements of H & N . Set $e = (e_2, e_1)$. Then

$$(e_2, e_1) (n, h) = (e_2 \cdot \boxed{\alpha(e_1)(n)}, e_1 h) = (n, h)$$

$\alpha(e_1) = \text{identity}$
 $N \rightarrow N$

$$(n, h) (e_2, e_1) = (n \cdot \boxed{\alpha(h)(e_2)}, h e_1) = (n, h)$$

$\alpha(h)$ is a gp. hom
so $\alpha(h)(e_2) = e_2$

(iii) Claim Inverse of $(n, h) = (\alpha(h)^{-1}(n^{-1}), h^{-1})$

Proof.

$$\begin{aligned} (n, h) \cdot (\alpha(h)^{-1}(n^{-1}), h^{-1}) &= (n \cdot \alpha(h)(\alpha(h)^{-1}(n^{-1})), h h^{-1}) \\ &= (n \cdot \alpha(e_1)(n^{-1}), e_1) = (e_2, e_1) = e \checkmark \end{aligned}$$

$$\begin{aligned} (\alpha(h)^{-1}(n^{-1}), h^{-1}) \cdot (n, h) &= ([\alpha(h)^{-1}(n^{-1})] \cdot [\alpha(h)^{-1}(n)], h^{-1} h) \\ &= (\alpha(h)^{-1}(e_2), e_1) = (e_2, e_1) = e \checkmark \end{aligned}$$

□

(19.1) Again, let $\boxed{\begin{matrix} H, N, \alpha: H \longrightarrow \text{Aut}(N) \\ \psi \\ e_1, e_2 \end{matrix}}$ identity elts. \longrightarrow gp

be given and $G = N \rtimes_{\alpha} H$ the group obtained in the lemma above.

Prop. $\begin{matrix} H & \longrightarrow & G \\ \psi & & \psi \\ h & \longmapsto & (e_2, h) \end{matrix}$ $\begin{matrix} N & \longrightarrow & G \\ \psi & & \psi \\ n & \longmapsto & (n, e_1) \end{matrix}$

are injective group homomorphisms, allowing us to view H & N as subgroups of G .

- (i) $H \leq G$; $N \trianglelefteq G$
- (ii) $N \cdot H = G$
- (iii) $H \cap N = \{e = (e_2, e_1)\}$

(i.e. G is a semidirect product of H & N .)

Proof. - $(e_2, h_1) (e_2, h_2) = (e_2 \cdot \alpha(h_1)(e_2), h_1 h_2)$
 $= (e_2, h_1 h_2)$

$\implies \begin{matrix} H & \longrightarrow & G \\ \psi & & \psi \\ h & \longmapsto & (e_2, h) \end{matrix}$ is a group hom.

Similarly, $(n_1, e_1) \cdot (n_2, e_2) = (n_1 \cdot (\alpha(e_1)(n_2)), e_1)$
 $= (n_1 n_2, e_1)$

\Rightarrow

$$\begin{array}{ccc} N & \longrightarrow & G \\ \downarrow & & \downarrow \\ n & \longrightarrow & (n, e_1) \end{array}$$

is a group hom. (injectivity of both is clear.)

Also, $(e_2, h) \cdot (n, e_1) \cdot (e_2, h)^{-1}$
 $= (e_2 \cdot \alpha(h)(n), h) \cdot (e_2, h^{-1})$
 $= (\alpha(h)(n), e_1) \in N$

$\Rightarrow N \trianglelefteq G.$

(i) is thereby proved. (ii) is true by definition of G .
 & (iii) are □

(19.2) Conversely, let G be a group which is a semidirect product of two subgroups H & N .

(i.e. $H \leq G$; $HN = NH = G$; $H \cap N = \{e\}$
 $N \trianglelefteq G$; \uparrow
 $\neq G$)

Let $\alpha : H \longrightarrow \text{Aut}_{gp}(N)$ be defined by

$$\alpha(h)(n) = \underbrace{h \cdot n \cdot h^{-1}}_{\substack{\text{group operation of} \\ \mathcal{G}}} \in N.$$

Take $G = N \rtimes_{\alpha} H$ as defined in Lemma on page 1.

Proposition. - $f : G \xrightarrow{\sim} \mathcal{G}$
 $(n, h) \longmapsto n \cdot h$ gp. operation of \mathcal{G} .

Proof. - f is a group hom :

$$f \left(\underbrace{(n_1, h_1) \cdot (n_2, h_2)}_{\substack{\text{of } G \text{ as defined} \\ \text{on page 1}}} \right) \stackrel{?}{=} f(n_1, h_1) \cdot \underbrace{f(n_2, h_2)}_{\text{of } \mathcal{G}}$$

$$\begin{aligned} \text{L.H.S.} &= f \left((n_1 \alpha(h_1)(n_2), h_1 h_2) \right) = n_1 \alpha(h_1)(n_2) h_1 h_2 \\ &= n_1 (h_1 n_2 h_1^{-1}) h_1 h_2 = n_1 h_1 n_2 h_2 \end{aligned}$$

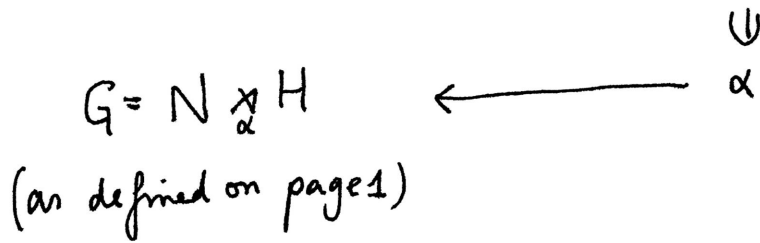
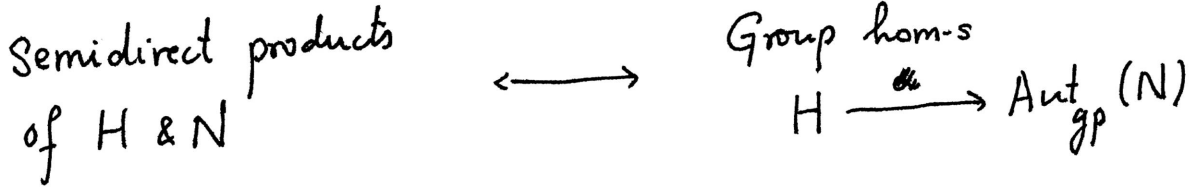
$$\text{R.H.S.} = n_1 h_1 n_2 h_2 \quad \checkmark$$

f is surjective because $N \cdot H = \mathcal{G}$.

f is injective because $N \cap H = \{e\}$ of G .

□

(19.3) Summary :



(19.4) Corollary of 3rd iso. thm. (consider the case when $G = H \cdot N$ and $H \cap N = \{e\}$.)

Let G be a semidirect product of H & N . Then, under natural projection $\pi : G \longrightarrow G/N$, restricted to H ,

$$\begin{array}{ccc} \psi & & \psi \\ \downarrow & & \downarrow \\ \pi & \longrightarrow & \pi \cdot N \end{array}$$

we obtain an iso. $\pi|_H : H \xrightarrow{\cong} G/N$.

$$\begin{array}{ccc} \psi & & \psi \\ \downarrow & & \downarrow \\ h & \longrightarrow & h \cdot N \end{array}$$

(read: π restricted to H) \nearrow

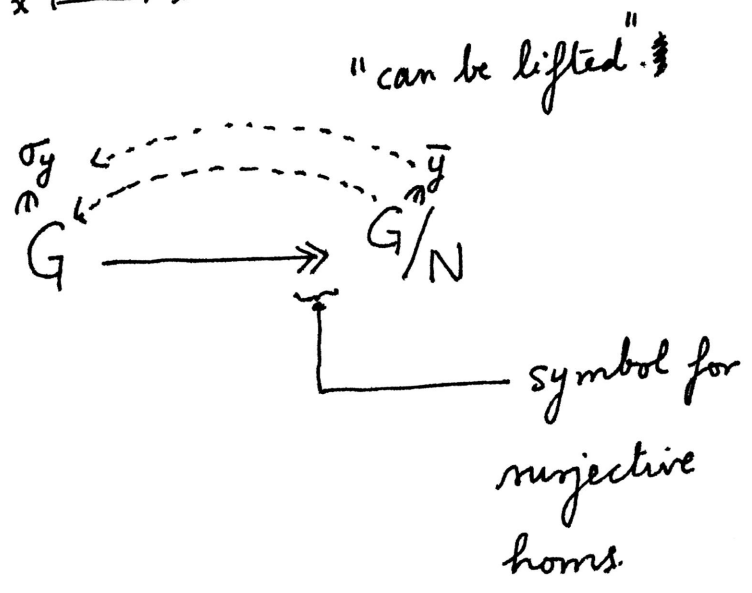
Verbal meaning. - for every coset $\bar{y} \in G/N$, we can find a representative $\bar{y} = \sigma_y N$ (of the same order as \bar{y} .)
 (take $\sigma_y = (\pi|_H)^{-1}(y) \in H \subset G$.)
 such that $\sigma_{y_1 y_2} = \sigma_{y_1} \sigma_{y_2}$.
 (order (σ) in G = order (\bar{y}) in G/N .)

A nice cartoon summarizing this observation obtained

directly from 3rd iso. thm $\left(\begin{array}{c} H \leq G; N \leq G \\ \downarrow \\ H/H \cap N \xrightarrow{\sim} H \cdot N/N \\ x \longmapsto xN \end{array} \right)$ and defn. of

semidirect product :

$$N \trianglelefteq G$$



(19.5) Aut_{gp}(W) = ? W: a group (finite mostly)

Example 1. Aut_{gp}(Z) has 2 elements:

iso 1. $1 \mapsto 1$ so identity

iso 2. $1 \mapsto -1$
 not identity, but squares to id.

$$\Rightarrow \text{Aut}_{gp}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

" $\{1, \sigma\}$