Lecture 19

(19.0) Constructing semidirect products.

Assume we are given two groups H and N; and a group homomorphism $\alpha : H \longrightarrow \mathrm{Aut}_{\text{group}}(N)$

$\uparrow$ set of all group $\underline{\text{iso}}$ $N \xrightarrow{\sim} N$

[Meaning: for every $h \in H$, $\alpha(h) : N \longrightarrow N$ is an isomorphism of N with itself. $\left(\alpha(h)^{-1} = \alpha(h^{-1})\right.$ see below.)

$$\alpha(e_1) = \text{Identity } \begin{array}{ccc} N & \longrightarrow & N \\ \Psi & & \Psi \\ n & \longmapsto & n \end{array}$$

e_1: unit of H

and $\alpha(h_1)\left(\alpha(h_2)(n)\right) = \alpha(h_1 h_2)(n)$.]

We can define a binary operation (2 inputs; 1 output) on $G = \{(n, h) \text{ where } n \in N \text{ and } h \in H\}$ as follows:

$$\boxed{(n_1, h_1) \cdot (n_2, h_2) = \left(n_1 \cdot \alpha(h_1)(n_2),\ h_1 h_2\right)} \quad \forall \begin{array}{l} n_1, n_2 \in N \\ h_1, h_2 \in H. \end{array}$$

Lemma. - The operation written above defines a group structure on the set $G = N \times H$.

Let us denote this group by $N \rtimes_\alpha H$.

this group   this group acts on

Proof. We need to check that (i) the operation defined above is associative; (ii) there is a neutral element and (iii) every element has an inverse.

(i). $g_1 = (n_1, h_1)$ $g_2 = (n_2, h_2)$ $g_3 = (n_3, h_3)$

(three typical elements of $G = N \times H$ - cartesian product of sets)

$$(g_1 \cdot g_2) \cdot g_3 = \left(n_1 \cdot [\alpha(h_1)(n_2)] \cdot [\alpha(h_1 h_2)(n_3)] \,,\, h_1 h_2 h_3\right) \quad \updownarrow \checkmark$$

$$g_1 \cdot (g_2 \cdot g_3) = \left(n_1 \cdot \left[\alpha(h_1)\left(n_2 \cdot [\alpha(h_2)(n_3)]\right)\right] \,,\, h_1 h_2 h_3\right)$$

$$\alpha(h_1)\left(n_2 \cdot \alpha(h_2)(n_3)\right) = \alpha(h_1)(n_2) \cdot \alpha(h_1)\left(\alpha(h_2)(n_3)\right)$$

$\alpha(h_1)$ is a gp. hom.

$$= \alpha(h_1)(n_2) \cdot \alpha(h_1 h_2)(n_3) \quad \checkmark$$

$\alpha : H \to \operatorname{Aut}_{gp.} N$ is a gp. hom

$$\Rightarrow (g_1 g_2) g_3 = g_1 (g_2 g_3).$$

(ii) Let $e_1 \in H$, $e_2 \in N$ be the respective identity elements of H & N. Set $e = (e_2, e_1)$. Then

$$(e_2, e_1)(n, h) = (e_2 \cdot \boxed{\alpha(e_1)(n)}, e_1 h) = (n, h)$$

$\alpha(e_1) = \text{Identity}$
$N \to N$

$$(n, h)(e_2, e_1) = (n \cdot \boxed{\alpha(h)(e_2)}, h e_1) = (n, h)$$

$\alpha(h)$ is a gp. hom
so $\alpha(h)(e_2) = e_2$

(iii) <u>Claim</u> Inverse of $(n, h) = (\alpha(h)^{-1}(n^{-1}), h^{-1})$

Proof.
$$(n, h) \cdot (\alpha(h)^{-1}(n^{-1}), h^{-1})$$
$$= \left(n \cdot \alpha(h)\left(\alpha(h^{-1})(n^{-1})\right), hh^{-1}\right)$$
$$= (n \cdot \alpha(e_1)(n^{-1}), e_1) = (e_2, e_1) = e \checkmark$$

$$(\alpha(h)^{-1}(n^{-1}), h^{-1}) \cdot (n, h)$$
$$= \left(\left[\alpha(h)^{-1}(n^{-1})\right] \cdot \left[\alpha(h)^{-1}(n)\right], h^{-1}h\right)$$
$$= (\alpha(h)^{-1}(e_2), e_1) = (e_2, e_1) = e. \checkmark$$

$\square$

(19.1) Again, let $\boxed{\begin{array}{ccc} H, & N, & \alpha : H \longrightarrow \mathrm{Aut}_{gp}(N) \\ \Psi & \Psi & \\ e_1 & e_2 & \end{array}}$

identity elts. $\longrightarrow e_1$

be given and $G = N \rtimes_\alpha H$ the group obtained in the lemma above.

Prop.

$$\begin{array}{ccc} H & \longrightarrow & G \\ \Psi & & \Psi \\ h & \longmapsto & (e_2, h) \end{array} \qquad \begin{array}{ccc} N & \longrightarrow & G \\ \Psi & & \Psi \\ n & \longmapsto & (n, e_1) \end{array}$$

are injective group homomorphisms, allowing us to view H & N as a subgroups of G.

(i) $H \leq G$; $N \trianglelefteq G$

(ii) $N \cdot H = G$

(iii) $H \cap N = \{e = (e_2, e_1)\}$

(i.e. G is a semidirect product of H & N.)

Proof. –

$$(e_2, h_1)(e_2, h_2) = (e_2 \cdot \alpha(h_1)(e_2) ; h_1 h_2)$$
$$= (e_2, h_1 h_2)$$

$$\Longrightarrow \quad \begin{array}{ccc} H & \longrightarrow & G \\ \Psi & & \Psi \\ h & \longmapsto & (e_2, h) \end{array} \quad \text{is a group hom.}$$

Similarly, $(n_1, e_1)\cdot(n_2, e_1) = \big(n_1\cdot(\alpha(e_1)(n_2)), e_1\big)$

$$= (n_1 n_2, e_1)$$

$$\Rightarrow \quad \begin{array}{ccc} N & \longrightarrow & G \\ \Psi & & \Psi \\ n & \longmapsto & (n, e_1) \end{array} \quad \text{is a group hom.}$$

(injectivity of both is clear.)

Also, $$\left.\begin{aligned} &(e_2, h)\cdot(n, e_1)\cdot(e_2, h)^{-1} \\ &= (e_2\cdot\alpha(h)(n), h)\,(e_2, h^{-1}) \\ &= (\alpha(h)(n), e_1) \in N \end{aligned}\right\} \Rightarrow N \trianglelefteq G.$$

(i) is thereby proved. (ii) & (iii) ~~is~~ are true by definition of G.

$\square$

(19.2) Conversely, let $\mathcal{G}$ be a group which is a semidirect product of two subgroups H & N.

(i.e. $H \leq \mathcal{G}$, $N \trianglelefteq \mathcal{G}$; $HN = NH = \mathcal{G}$; $H \cap N = \{e\}$ of $\mathcal{G}$)

Let $\alpha : H \longrightarrow \mathrm{Aut}_{gp}(N)$ be defined by

$$\alpha(h)(n) = h \cdot n \cdot h^{-1} \in N.$$

group operation of $\mathcal{G}$

Take $G = N \rtimes_{\alpha} H$ as defined in Lemma on page 1.

Proposition. – $f : G \xrightarrow{\sim} \mathcal{G}$

$$(n, h) \longmapsto n \cdot h$$

gp. operation of $\mathcal{G}$.

Proof. – <u>f is a group hom</u> :

$$f\left((n_1, h_1) \cdot (n_2, h_2)\right) \stackrel{?}{=} f(n_1, h_1) \cdot f(n_2, h_2)$$

of G as defined on page 1

of $\mathcal{G}$

$$\text{L.H.S.} = f\left((n_1\, \alpha(h_1)(n_2),\, h_1 h_2)\right) = n_1\, \alpha(h_1)(n_2)\, h_1 h_2$$

$$= n_1 \left(h_1 n_2 h_1^{-1}\right) h_1 h_2 = n_1 h_1 n_2 h_2$$

$$\text{R.H.S} = n_1 h_1 n_2 h_2 \quad \checkmark$$

f is surjective because $N \cdot H = \mathcal{G}$.

f is injective because $N \cap H = \underbrace{\{e\}}_{\text{of } G}$

$\square$

(19.3) Summary:

$$\begin{array}{ccc} \text{Semidirect products of } H \,\&\, N & \longleftrightarrow & \text{Group hom-s } H \xrightarrow{\alpha} \mathrm{Aut}_{gp}(N) \\ & & \cup\!\!\!\!| \\ G = N \rtimes_{\alpha} H & \longleftarrow & \alpha \end{array}$$

(as defined on page 1)

(19.4) Corollary of 3rd iso. thm. (consider the case when $G = H \cdot N$ and $H \cap N = \{e\}$.)

Let G be a semidirect product of H & N. Then, under natural projection

$$\begin{array}{rccc} \pi : & G & \longrightarrow & G/N \\ & \cup\!\!\!\!| & & \cup\!\!\!\!| \\ & x & \longmapsto & x \cdot N \end{array}$$

, restricted to H,

we obtain an iso. $$\begin{array}{rccc} \pi\big|_H : & H & \xrightarrow{\cong} & G/N \\ & \cup\!\!\!\!| & & \cup\!\!\!\!| \\ & h & \longmapsto & hN \end{array}$$

(read: π restricted to H)

Verbal meaning.- for every cost $\bar{y} \in G/N$, we can find a representative $\bar{y} = \sigma_y N$ $(\sigma \in G)$ (of the same order as $\bar{y}$). (take $\sigma_y = (\pi|_H)^{-1}(y) \in H \subset G$.)

such that $\sigma_{y_1 y_2} = \sigma_{y_1} \sigma_{y_2}$.

$\left(\underset{\text{in } G}{\mathrm{order}}(\sigma) = \underset{\text{in } G/N}{\mathrm{order}}(\bar{y})\right)$.

A nice cartoon summarizing this observation obtained directly from 3^{rd} iso. thm $\left(\begin{array}{c} H \leq G;\ N \trianglelefteq G \\ \Downarrow \\ H/_{H\cap N} \xrightarrow{\simeq} H.N/_N \\ x \longmapsto xN \end{array}\right)$ and defn. of semidirect product :

"can be lifted".

$$N \trianglelefteq G \qquad \begin{tikzcd} \sigma_y & & \bar{y} \arrow[ll, dashed, bend right] \\ G \arrow[rr, two heads] & & G/_N \arrow[ll, dashed, bend right] \end{tikzcd}$$

symbol for surjective homs.

(19.5) $\underline{\mathrm{Aut}_{gp}(W) = ?}$ W: a group (finite mostly)

Example 1. $\mathrm{Aut}_{gp}(\mathbb{Z})$ has 2 elements : Iso 1. $1 \longmapsto 1$ so identity

Iso 2. $\underbrace{1 \longmapsto -1}$

not identity, but squares to id.

$$\Rightarrow \mathrm{Aut}_{gp}(\mathbb{Z}) \cong \mathbb{Z}/_{2\mathbb{Z}}$$
$$\| $$
$$\{1, \sigma\}$$