Lecture 24

(24.0) Recall the "plan to study finite groups" –

Classify all simple groups! ← NOT part of (any?) course.

General results on <u>how they fit together</u>.

Definition. – Let G be a group. A <u>composition series</u> of G is a <u>finite sequence of normal subgroups</u>.

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \quad \cdots \quad \trianglerighteq G_n = \{e\}$$

Let Σ denote this finite sequence of normal subgroups.

$\left\{ G_i / G_{i+1} \right\}_{i=0,1,2,\ldots,n-1}$ are often called <u>graded pieces of Σ</u>. (or G via Σ)

Sometimes we will use the notation $gr_i^{\Sigma}(G)$ to denote "i^{th} graded piece of G obtained via composition series Σ"

$= G_i / G_{i+1}$. We say Σ is <u>strict</u>; if $G_i \underset{\neq}{\triangleright} G_{i+1}$ $\forall\ i = 0, \ldots, n-1$. (i.e. $gr_i^{\Sigma}(G) \neq \{e\}$).

(24.1) Example : $G = D_{2n}$ dihedral group.

$\Sigma_1 : D_{2n} = G_0 \trianglerighteq G_1 = \langle r \rangle \cong \mathbb{Z}/n\mathbb{Z} \trianglerighteq \{e\}$ a composition series (strict).

can put more terms here. "refinement".

Graded pieces $\{ \mathbb{Z}/2\mathbb{Z} \,;\, \mathbb{Z}/n\mathbb{Z} \}$.

Example : $S_n \trianglerighteq A_n \trianglerighteq \{e\}$ a strict composition series for S_n.

cannot be "refined"

Graded pieces : $\{ \mathbb{Z}/2\mathbb{Z} \,;\, A_n \}$

(24.2) Let G be a group and let there be two composition series

$$\Sigma_1 : G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

$$\Sigma_2 : G = G_0' \trianglerighteq G_1' \trianglerighteq \cdots \trianglerighteq G_m' = \{e\}.$$

We say Σ_2 is <u>finer than</u> Σ_1 (or, a <u>refinement</u> of Σ_1) if Σ_1 is obtained from Σ_2 by omitting a few terms.

Warning: $A \trianglelefteq B \trianglelefteq C$ does <u>not</u> imply $A \trianglelefteq C$

(e.g. Take any non-trivial group H and form a semidirect product

$$(C =)\ G = (H \times H) \rtimes_\alpha \underset{\{e, s\}}{\overset{}{\mathbb{Z}/2\mathbb{Z}}} \qquad \left(\alpha(s) : (x,y) \mapsto (y,x)\right)$$

$$(B =)\quad H \times H$$

$$(A =)\quad H \ (\text{say } 1^{st} \text{ component})$$

with $G \trianglerighteq H \times H \trianglerighteq H$; H is not normal in G here ($s H_{1^{st}\text{ component}} s^{-1} = H_{2^{nd}\text{ comp.}}$)).

So we <u>can not</u> arbitrary drop terms from a composition series and still get a composition series (in general).

e.g.

$$G_0 = D_{12} \trianglerighteq G_1 = \langle r \rangle \trianglerighteq \{e\} \ : \Sigma_1$$
$$(\cong \mathbb{Z}/6\mathbb{Z})$$

$$G_0' = D_{12} \trianglerighteq G_1' = \langle r \rangle \trianglerighteq G_2' = \langle r^2 \rangle \trianglerighteq \{e\} : \Sigma_2$$
$$(\cong \mathbb{Z}/6\mathbb{Z}) \qquad (\cong \mathbb{Z}/3\mathbb{Z})$$

Σ_2 is finer than Σ_1 (or Σ_1 is a "coarsening of Σ_2" i.e. some terms dropped.)

(24.3) Let G and H be two groups and let

$$\Sigma : \quad G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

$$\Sigma' : \quad H = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_m = \{e\},$$

be two composition series. We say Σ and Σ' are <u>equivalent</u> if $m = n$ and we have a permutation $\sigma \in S_n$ such that $G_i / G_{i+1} \cong H_{\sigma(i)} / H_{\sigma(i)+1}$.

S_n: permutations of $\{0, 1, \dots, n-1\}$

(i.e. the sets $\{G_i / G_{i+1}\}_{0 \leq i \leq n-1}$ and $\{H_j / H_{j+1}\}_{0 \leq j \leq m-1}$ are "<u>the same</u>".)

Note : we didn't ask that G & H be isomorphic. In fact they need not be, as the next example shows.

(24.4) $G = D_8$ with the following composition series:

$$\underset{\Sigma}{D_8 = G = G_0} \trianglerighteq G_1 = \langle r \rangle \cong \mathbb{Z}/4\mathbb{Z} \trianglerighteq G_2 = \langle r^2 \rangle \trianglerighteq \{e\} = G_3$$
$$(\cong \mathbb{Z}/2\mathbb{Z})$$

Take $H = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

$$\left(\text{group operation} \quad \begin{array}{ll} i^2 = j^2 = k^2 = -1 & (-1)^2 = 1. \\ i \cdot j = -j \cdot i = k & \\ j \cdot k = -k \cdot j = i & \\ k \cdot i = -i \cdot k = j & \end{array} \right)$$

$$\underset{\Sigma'}{Q_8 = H = H_0} \trianglerighteq H_1 = \{\pm 1, \pm i\} \trianglerighteq H_2 = \{\pm 1\} \trianglerighteq \{e\} = H_3$$
$$(\cancel{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}) \qquad (\cong \mathbb{Z}/2\mathbb{Z})$$

We get same graded pieces ($\mathbb{Z}/2\mathbb{Z}$ 3 times).

So Σ and Σ' are equivalent. But

$$D_8 \not\cong Q_8$$

elements of order 4 : $\quad 2 \quad\quad 6$

$$\underset{r,\, r^3}{\overset{2}{\uparrow}} \qquad \underset{\pm i,\, \pm j,\, \pm k.}{\overset{6}{\uparrow}}$$

(24.5) A composition series Σ of G is said to be a <u>Jordan-Hölder</u> series if it is <u>strict</u> and any strict composition series Σ' finer than Σ is <u>equal</u> <u>to</u> Σ.

(Jordan-Hölder = Maximal among all strict composition series.)

Note: if $|G| = \infty$, it may not <u>admit any</u> Jordan-Hölder series.

eg. $\mathbb{Z} = G = G_0$. As

$$\mathbb{Z} \trianglerighteq 2\cdot\mathbb{Z} \trianglerighteq 2^2\cdot\mathbb{Z} \trianglerighteq 2^3\cdot\mathbb{Z} \trianglerighteq \cdots$$

if stopped, can always be further refined.

Lemma. Every <u>finite G</u> admits (at least) one Jordan-Hölder series.

Proof. – Choose a <u>max'l</u> normal proper subgroup of G, say N. (Meaning – say, e.g., $|N|$ = max among all proper normal subgroups of G.).

If $N = \{e\}$, G is simple and hence the only composition series, <u>strict</u>, for G is $G \trianglerighteq \{e\}$.

Otherwise $G = G_0 \ntrianglerighteq N \ntrianglerighteq \{e\}$ is a strict composition series and G/N is simple, because

(2nd iso. thm.) Normal Subgps. of G/N $\leftrightarrow$ Normal Subgroups of G containing N.

(& $N \trianglelefteq G$ was maximal.)

Continuing this way, we obtain

$$(*) - \quad G = G_0 \ntrianglerighteq G_1 \ntrianglerighteq G_2 \ntrianglerighteq \cdots \ntrianglerighteq G_n = \{e\}$$

s.t. each G_i/G_{i+1} is simple.

It remains to check that $(*)$ cannot be further refined (i.e., is a Jordan-Hölder series).

If $(*)$ can be refined by a strict composition series, say,

$$\Sigma' : \quad G = G_0' \underset{\neq}{\trianglerighteq} G_1' \underset{\neq}{\trianglerighteq} \cdots \underset{\neq}{\trianglerighteq} G_m' = \{e\}$$

then we can find smallest $i \geq 0$ s.t.

$$\begin{array}{ccc} G_i & = & G_i' \\ \neq\,\triangledown & & \neq\,\triangledown \\ G_{i+1} & & G_{i+1}' \\ & & \vdots \\ & & \neq\,\triangledown \\ & & G_{i+r}' \end{array} \qquad G_{i+1} = G_{i+r}'$$

(i.e. Σ is obtained from Σ' by omitting & may be more.)

But there are no normal subgroups between G_i & G_{i+1} since

G_i / G_{i+1} is simple.