Lecture 25

(25.0) Recall that for a group G, a composition series Σ is just a descending sequence of normal subgroups

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

(remember G_{j+2} <u>need not be normal</u> in G_j.)

- Graded pieces associated to Σ : $gr_i^{\Sigma}(G) = G_i / G_{i+1}$ $(i = 0, 1, \ldots, n-1)$
- Length of Σ $= n$
- Σ' is a refinement of Σ if Σ is obtained from Σ' by omitting a few terms (in particular, $\text{length}(\Sigma) \leq \text{length}(\Sigma')$)
- Σ_1 is equivalent to Σ_2 if (i) they have same lengths (ii) they have same associated graded pieces (possibly up to permutation.)

e.g.

$$\left.\begin{array}{l} \mathbb{Z}/4\mathbb{Z} \trianglerighteq \mathbb{Z}/2\mathbb{Z} \trianglerighteq \{0\} \\[2em] (\mathbb{Z}/2\mathbb{Z})^2 \trianglerighteq \underset{(\text{say } 1^{st} \text{ component})}{\mathbb{Z}/2\mathbb{Z}} \trianglerighteq \{0\} \end{array}\right] \text{ are equivalent.}$$

- Σ is <u>strict</u> if $gr_i^{\Sigma}(G) \neq \{e\}$ $(\forall i=0,\dots,n-1)$

  $(\equiv \quad G_i \ntrianglerighteq G_{i+1})$

- <u>Jordan-Hölder</u> if <u>strict</u> and "<u>maximal</u> relative to refinement (among all strict composition series)". Meaning:

Σ is J-H if strict & <u>for every Σ' strict and finer than Σ ; $\Sigma' = \Sigma$</u>.

(25.1) In Lecture 24 (page 6), we showed that every finite group admits a Jordan-Hölder series. We notice, from the proof of Lemma (24.5) - (pages 6-8), that ~~all composition~~

$$\boxed{\Sigma \text{ is Jordan-Hölder} \iff gr_i^{\Sigma}(G) \text{ is simple } \quad i=0,\dots,n-1}$$

We ~~are~~ still don't know if Jordan-Hölder series are unique. This is the goal of today's lecture. We will prove:

(25.2) Theorem: [Schrier] Let Σ_1 and Σ_2 be two composition series of a group G. Then there exist refinements Σ_1', Σ_2' of Σ_1, Σ_2 respectively, such that Σ_1' is equivalent to Σ_2'.

Let us first see why this theorem implies the uniqueness of Jordan-Hölder series. Let Σ_1 and Σ_2 be two J-H series of a group G. The theorem produces refinements:

$$\begin{cases} \Sigma_1' \text{ finer than } \Sigma_1 \\ \Sigma_2' \quad \text{"} \quad \text{"} \quad \Sigma_2 \end{cases} ; \quad \Sigma_1' \text{ equivalent to } \Sigma_2'.$$

But Σ_1 (& Σ_2) is maximal among <u>strict</u> composition series. That means Σ_1' (& Σ_2') cannot be strict. That is, if

$$\Sigma_1 : \quad G = G_0 \underset{\neq}{\triangleright} G_1 \underset{\neq}{\triangleright} \cdots \underset{\neq}{\triangleright} G_n = \{e\}$$

then, necessarily, Σ_1' has the form

$$\Sigma_1' : \quad G_0 = \underbrace{\boxed{G_0 = G_0 \cdots = G_0}}_{\text{repeated } k_0 \text{ times}} \underset{\neq}{\triangleright} \overbrace{\boxed{G_1 = \cdots = G_1}}^{\text{repeated } k_1 \text{ times}} \underset{\neq}{\triangleright} \cdots$$

and so on...

Same conclusion applies to Σ_2'. But then the equivalence between Σ_1' and Σ_2' implies that <u>Σ_1 & Σ_2 are equivalent</u>

(Ex. see why? - we just have to omit $\{e\}$ from both (bunch of times) $\{gr_j^{\Sigma_1'}(G)\}$ to get $\{gr_i^{\Sigma_1}(G)\}$.)

We have thus proved:

(25.3) Corollary. – If Σ_1, Σ_2 are two Jordan-Hölder Series of G, then Σ_1 and Σ_2 are equivalent (i.e., have same length and, <u>up to relabelling</u>, same associated graded pieces).

e.g. $\Sigma_1 : \mathbb{Z}/6\mathbb{Z} \underset{\neq}{\trianglerighteq} \mathbb{Z}/3\mathbb{Z} \underset{\neq}{\trianglerighteq} \{0\}$

$\Sigma_2 : \mathbb{Z}/6\mathbb{Z} \underset{\neq}{\trianglerighteq} \mathbb{Z}/2\mathbb{Z} \underset{\neq}{\trianglerighteq} \{0\}$

both J-H series

Associated graded with respect to Σ_1 : $\{\overset{0^{th}}{\mathbb{Z}/2\mathbb{Z}} ; \overset{1^{st}}{\mathbb{Z}/3\mathbb{Z}}\}$ ✓

" " " " " Σ_2 : $\{\mathbb{Z}/3\mathbb{Z} ; \mathbb{Z}/2\mathbb{Z}\}$

(25.4) <u>Proof of Theorem (25.2)</u> : Assume we are given two composition series Σ_1, Σ_2 of a group G.

$\Sigma_1 : \; G = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_n = \{e\}$

$\Sigma_2 : \; G = K_0 \trianglerighteq K_1 \trianglerighteq \cdots \trianglerighteq K_m = \{e\}$

Idea: for each $i = 0, \dots, n-1$; zoom in between

$$H_i \trianglerighteq H_{i+1}$$

$$K_0 \trianglerighteq K_1 \trianglerighteq \dots \triangleright K_m$$

$$\rightsquigarrow \quad H_i = (H_i \cap K_0) \cdot H_{i+1} \trianglerighteq (H_i \cap K_1) \cdot H_{i+1} \trianglerighteq \dots \triangleright H_{i+1} = (H_i \cap K_m) \cdot H_{i+1}$$

total $(m+1)$ terms

Normality is shown on page 7 below

Thus we form Σ_1' (using both Σ_1 & Σ_2) as follows:

$$\text{Set } \mathcal{L}_{i;j} = (H_i \cap K_j) \cdot H_{i+1} \quad \begin{pmatrix} 0 \le i \le n-1 \\ 0 \le j \le m \end{pmatrix}$$

$$G = \mathcal{L}_{0;0} \trianglerighteq \mathcal{L}_{0;1} \trianglerighteq \dots \trianglerighteq \mathcal{L}_{0;m} = H_1$$

$$(H_0 = K_0)$$

$$H_1 = \mathcal{L}_{1;0} \trianglerighteq \mathcal{L}_{1;1} \trianglerighteq \dots \trianglerighteq \mathcal{L}_{1;m} = H_2$$

$$H_2 = \mathcal{L}_{2;0} \quad \dots$$

$$\vdots$$

$$H_{n-1} = \mathcal{L}_{n-1;0} \trianglerighteq \dots \trianglerighteq \mathcal{L}_{n-1;m} = H_n = \{e\}$$

[Fig: how Σ_1' looks: & why it refines Σ_1.]

Σ_1 obtained from Σ_1' by omitting many terms

Length of $\Sigma_1' = m \cdot n$

Similarly we can define Σ_2' - a refinement of Σ_2, using Σ_1. Let $R_{ij} = (H_i \cap K_j) \cdot K_{j+1}$

$0 \leq i \leq n$
$0 \leq j \leq m-1$

Now i increases first!

$$K_0 = R_{0;0} \trianglerighteq R_{1;0} \trianglerighteq \cdots \trianglerighteq R_{m;0} = K_1$$

- - - and so on. <u>Length (Σ') is still $m.n$.</u>

Thus to prove that Σ_1' and Σ_2' are equivalent, we need to identify their associated graded pieces -

$$\mathcal{L}_{i;j} \trianglerighteq \mathcal{L}_{i;j+1} \qquad\qquad R_{i;j} \trianglerighteq R_{i+1;j}$$

In Σ_1' $\qquad\qquad$ in Σ_2'

same quotient - (to prove yet!)

There are only 4 relevant subgroups of G in the discussion:

$$\mathcal{L}_{ij} = (H_i \cap K_j) \cdot H_{i+1} \qquad R_{ij} = (H_i \cap K_j) \cdot K_{j+1}$$

$$\mathcal{L}_{i;j+1} = (H_i \cap K_{j+1}) H_{i+1} \qquad R_{i+1;j} = (H_{i+1} \cap K_j) \cdot K_{j+1}$$

Set $H = H_i \trianglerighteq H' = H_{i+1}$
$K = K_j \trianglerighteq K' = K_{j+1}$ for notational convenience only.

and the next result finishes the proof of our theorem by showing (1) $\mathcal{L}_{i;j+1}$ is normal in $\mathcal{L}_{i;j}$ (similarly, $\mathcal{R}_{i+1;j}$ ——— $\mathcal{R}_{i;j}$

(2) $$\mathcal{L}_{i;j}\big/\mathcal{L}_{i;j+1} \cong \mathcal{R}_{i;j}\big/\mathcal{R}_{i+1;j}$$

(25.5) Zassenhaus Lemma. Let H, K be two subgroups of G and $H' \trianglelefteq H$, $K' \trianglelefteq K$. Then

(i) $(H \cap K')\cdot H'$ is normal subgp. of $(H \cap K).H'$
$(K \cap H')\cdot K'$ ——— $(K \cap H).K'$

(ii) $$(H \cap K).H' \big/ (H \cap K').H' \cong (K \cap H).K' \big/ (K \cap H').K' \quad \text{—} (*)$$

[(i) is left as the following exercise :

$K \trianglerighteq K'$ (given) $\Rightarrow$ $H \cap K \trianglerighteq H \cap K'$. This and $H \trianglerighteq H'$ $\Rightarrow$ $(H \cap K)\cdot H' \trianglerighteq (H \cap K').H'$ $\left(\subseteq H\right)$.]

(ii) is proved by showing that

$$\frac{(H\cap K)\cdot H'}{(H\cap K')\cdot H'} \cong \frac{H\cap K}{(H\cap K')\cdot(H'\cap K)}$$

symmetric in H & K variables

(so the same is true for the R.H.S. of (*) in the statement of the lemma)

(Note: $H\cap K'$, $H'\cap K$ are both normal in $H\cap K$. Hence, so is $(H\cap K')\cdot(H'\cap K)$ – prove this!)

But this is precisely the "3rd iso. thm." – (Lecture 18)

$$\text{Subgp}\big/\text{subgp}\cap\text{normal} \;\cong\; \text{Subgp}\cdot\text{Normal}\big/\text{normal}$$

$\boxed{\cancel{\text{In } H \trianglerighteq H' \text{ (normal)}, \geq H\cap K \text{ (subgroup) and}}}$ In $(H\cap K)\cdot H'$ $\trianglerighteq$ $(H\cap K')\cdot H'$ (normal) $\geq$ $H\cap K$ (subgp)

we get

$$\frac{H\cap K}{\big((H\cap K')\cdot H'\big)\cap(H\cap K)} \cong \frac{(H\cap K)\cdot H'}{(H\cap K')\cdot H'}$$

It remains to check:

$$((H\cap K')\cdot H')\cap(H\cap K) = (H\cap K')\cdot(H'\cap K)$$

both are contained in $H\cap K$ as well as $(H\cap K)\cdot H'$ $\Rightarrow$ $\supseteq$ ✓

Now let $x = a\cdot b \in ((H\cap K')\cdot H')\cap(H\cap K)$

where $a\in H\cap K'$ and $b\in H'$.
(so $a\in H\cap K$)

We get $a^{-1}x \;(= b) \in H\cap K$ (because both $a, x\in H\cap K$)

i.e. $b\in H'\cap H\cap K = H'\cap K$.

$\Rightarrow\; x = a\cdot b \in (H\cap K')\cdot(H'\cap K)$. $\square$