Lecture 26

(26.0) Recall : we introduced the notion of a composition series of a group G.

- Composition series = descending chain of normal subgroups.

$$\Sigma : G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

- Jordan - Hölder series = strict composition series, maximal with respect to refinement.

Certain composition series can be constructed via "commutators".

(26.1) Definition. - Let G be a group and $x, y \in G$.

$$[x,y] := xyx^{-1}y^{-1} \in G.$$

Given $A, B \subset G$ (just subsets),

$$[A,B] := \text{subgroup of } G \text{ generated by } \left\{ [a,b] : \begin{matrix} a \in A \\ b \in B \end{matrix} \right\}.$$

Lemma. If $A, B \trianglelefteq G$, then $[A,B] \trianglelefteq G$.

(The proof is same as the one when $A = B = G$ see, e.g., Lecture 7 (page 3). We reproduce the argument below.)

Proof. Given $x = [a,b] = aba^{-1}b^{-1}$; $a \in A$ & $b \in B$,

and $g \in G$, we have :

$$g x g^{-1} = [\underbrace{gag^{-1}}_{\in A}, \underbrace{gbg^{-1}}_{\in B}] \in [A,B]$$ as A & B are <u>normal</u> in G.

In general; $x \in [A,B]$ is of the form $x = x_1 x_2 \cdots x_k$, where $x_j = [a_j, b_j]$ for some $a_j \in A$, $b_j \in B$ $(j = 1, 2, \ldots, k)$.

Again we obtain

$$g x g^{-1} = (g x_1 g^{-1})(g x_2 g^{-1}) \cdots (g x_k g^{-1}) \in [A,B].$$

Hence, $[A,B] \trianglelefteq G$. $\square$

(26.2) Commutator series (also known as "derived series").

[<u>caution</u>: the word "derived" has changed its meaning these days. We have things like "derived categories, functors, derived algebraic geometry ...". So, it is, perhaps, not a good idea to use it here. I will try to stick to "commutator series".]

Let G be a group. Take $G^{(0)} = G$,

$$G^{(1)} = [G, G] \; ; \quad G^{(2)} = [G^{(1)}, G^{(1)}] \; ; \quad \cdots$$

$$G^{(\ell+1)} = [G^{(\ell)}, G^{(\ell)}] \; ; \quad \cdots$$

We obtain $G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq \cdots$ (may not end! even if G is finite.)

called commutator series of G.

<u>Remark</u>. For any group H, the quotient by $[H,H] \trianglelefteq H$, $H/[H,H]$ is <u>abelian</u>. It is "the largest abelian group which admits a <u>surjective</u> group hom. <u>from</u> H."

That is, if A is any abelian group and $f : H \longrightarrow A$ is a gp. hom., then $[H,H] \subset \mathrm{Ker}(f)$. Moreover, for

$$f = \pi : H \longrightarrow H/[H,H] \, , \quad \mathrm{Ker}(\pi) = [H,H].$$

Thus, for the commutator series, the $\underbrace{\text{associated graded pieces}}_{\uparrow}$ $\left(G^{(0)}/G^{(1)} \; ; \; G^{(1)}/G^{(2)} \; ; \; \cdots \right)$ are <u>abelian</u>.

(26.3) Examples.

(i) G abelian $\Longrightarrow$ $G^{(0)} = G \trianglerighteq G^{(1)} = \{e\}$.

(ii) $G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, c \neq 0 ;\ b \text{ arbitrary} \right\}$

$$\left[\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \right] = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \begin{pmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} \begin{pmatrix} \frac{1}{a'} & \frac{-b'}{a'c'} \\ 0 & \frac{1}{c'} \end{pmatrix}$$

$$\left[\text{Note: } \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}^{-1} = \begin{bmatrix} \frac{1}{x} & \frac{-y}{xz} \\ 0 & \frac{1}{z} \end{bmatrix}. \right]$$

$$= \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

where $d = \dfrac{ab' - a'b + bc' - b'c}{cc'}$ (whatever!)

Then $G^{(1)}$ consist of $\left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \text{ arbitrary} \right\}$

↑ this group is abelian because $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+x' \\ 0 & 1 \end{pmatrix}$.

$\Rightarrow$ $G^{(2)} = \{e\}$. Thus the commutator series of this group is $G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} = \{e\}$.

(iii) Assume G is simple, non-abelian, say A_5.

$[G, G] \trianglelefteq G$ (by Lemma (26.1) page 1).

$[G,G] \neq \{e\}$ because G is not abelian

(not every $ab = ba$ i.e. not every $aba^{-1}b^{-1} = e$).

As G is simple, we conclude that $[G,G] = G$.

Hence $G^{(0)} = G = G^{(1)} = G^{(2)} = \dots$ (never ending!).

(26.4) Definition. — We say G is solvable if for some $n \geq 0$, we have $G^{(n)} = \{e\}$.

Examples (i) and (ii) above are solvable. A_n (or any simple group for that matter) are not solvable

(26.5) The term "solvable" originated from the concept of "solvability by radicals of a polynomial equations". You will learn more about it next term (Galois theory!), but here is the rough dictionary to illustrate this.

$ax^2 + bx + c = 0$ can be "solved" $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ e.g.

in terms of
- polynomials in $\{a,b,c\}$
- radicals (square-root)

The "reason" for this will turn out to be - $\mathbb{Z}/2\mathbb{Z} = S_2$ is solvable.

- $S_3 = G = G^{(0)} \trianglerighteq G^{(1)} \cong \mathbb{Z}/3\mathbb{Z} \trianglerighteq G^{(2)} = \{e\}$ also solvable

$\left(\overset{\shortparallel}{D_6} = \langle s, r \mid \dots \rangle \right)$

$r^{-1} = r^2 = srs \Rightarrow srs^{-1}r^{-1} = r \in G^{(1)}$

[Any degree 3 poly eq^n can be solved, ~~see solution~~ look up the general case]

e.g. ~~$x^3 + Ax + B = 0$~~ $x^3 - 3Ax + B = 0$ is "solved":

$$x = r + \frac{A}{r} \quad \text{where } r = \left[\frac{B \pm \sqrt{B^2 - 4A^3}}{2} \right]^{\frac{1}{3}}$$

radicals

Ex. S_4 is also solvable.

(read about solving degree 4 eq^n - if you are interested.)

S_5 (& S_n beyond; $n \geq 5$) are <u>not</u> <u>solvable</u>.

(You may already know this - a general degree 5 eq^n has no "explicit soln." like the ones above).