Lecture 27

(27.0) Recall: we defined <u>solvable groups</u> as follows

- Let $G^{(0)} = G$; $G^{(1)} = [G^{(0)}, G^{(0)}]$; $\cdots$;

$$G^{(\ell+1)} = [G^{(\ell)}, G^{(\ell)}] \quad (\trianglelefteq G^{(\ell)} \text{ by Lemma (26.1) of Lecture 26}).$$

- G is <u>solvable</u> if $G^{(N)} = \{e\}$ for sufficiently large N.

(27.1) Properties of commutator subgroup.

(i) $G/[G;G]$ is abelian (hence, <u>any subgroup</u> of G containing $[G;G]$ is <u>normal</u>. This is because if $[G;G] \subseteq H \leq G$ then $\pi(H)$ is normal in $G/[G;G]$, where $\pi : G \longrightarrow G/[G;G]$. But

$G/[G;G]$ ← abelian. So <u>any</u> subgroup is normal

$$\text{Normal Subgps of } G/[G;G] \leftrightarrow \text{\underline{Normal} subgps of } G \text{ containing } [G;G]).$$

(ii) Conversely, if $H \trianglelefteq G$ s.t. G/H is abelian, then $[G; G] \subset H$.

(27.2) Thus (as we already observed in (26.2)), $G^{(l)}/G^{(l+1)}$ is abelian for every $l \geq 0$. We have the following converse to this assertion.

Theorem. Let G be a group. The G is solvable if and only if there exists a composition series

$$\Sigma : \quad G = K_0 \trianglerighteq K_1 \trianglerighteq \cdots \trianglerighteq K_m = \{e\}$$

such that $gr_j^{\Sigma}(G) \left(= K_j/K_{j+1}\right)$ is abelian for every $j = 0, 1, \ldots, m$.

Proof. $(\Rightarrow)$ Assume G is solvable. Then $G^{(N)} = \{e\}$ for some $N \geq 0$. Take $K_j = G^{(j)}$ $(m = N)$.

$(\Leftarrow)$ We will prove that $G^{(j)} \subset K_j$ for every j, by induction. Base Case: $j = 0$: $G^{(0)} = K_0 = G$

Induction Hypothesis: $G^{(j)} \subset K_j$ for $j = 0, \dots, \ell$ (for some $\ell \lneq m$).

Induction Step. Since $K_\ell / K_{\ell+1}$ is abelian (given), we get

$[K_\ell ; K_\ell] \subset K_{\ell+1}$ (see (ii) on the top of last page).

$$\Rightarrow \underbrace{[G^{(\ell)} ; G^{(\ell)}]}_{= G^{(\ell+1)}} \underset{\text{Induction Hypothesis}}{\subset} [K_\ell ; K_\ell] \subset K_{\ell+1}.$$

Hence $G^{(m)} \subset K_m = \{e\} \Rightarrow G$ is solvable $\square$

(27.3) Consequences of Theorem (27.2).

1. $|G| = p^r \Rightarrow G$ is solvable.

Recall that we proved in Lecture 13 that <u>p-groups always have non-trivial center</u>. We obtained this result from a more general statement:

$$\boxed{G \circlearrowright X, \ |G| = p^r \Rightarrow |X| \equiv |\underbrace{X^G}_{\{x \in X \mid g \cdot x = x \ \forall g \in G\}}| \pmod p}$$

[Proof. $X =$ disjoint union of orbits under G-action.

\# elements in an orbit divides $|G| = p^r$ $(r \geq 1)$.

$\Rightarrow$ $|X| \equiv$ \# orbits that ~~are~~ just have one element (mod p)

$= |X^G|$ (mod p). $\square$

If we take $G \circlearrowright G$ by conjugation (i.e. $g \cdot x = g x g^{-1}$)

then we get $|G| \equiv \left| \{ x \in G \mid g x g^{-1} = x \ \forall g \in G \} \right|$ (mod p)

this set = center of G denoted by $Z(G)$. $\square$]

Proof of $\boxed{\text{Every } p\text{-group is solvable}}$:

(see also Proposition on page 3 of Lecture 13.)

Ex. $|G| = p^n \Rightarrow$ we have a chain of normal subgroups (i.e. composition series)

$$G = Z_0 \trianglerighteq Z_1 \trianglerighteq \cdots \trianglerighteq Z_r = \{e\}$$

such that Z_j / Z_{j+1} is abelian.

Hint: induction on n (when we write $|G| = p^n$).

$n=1 \Rightarrow G \cong \mathbb{Z}/p\mathbb{Z}$ abelian hence solvable.

$n>1$, replace G by $G/Z(G)$.

Now, by Theorem (27.2), we are done.

(27.4) Consequences of Thm (27.2).

2. Let G be a group and $N \trianglelefteq G$, be a normal subgroup. Then

G is solvable if and only if $\begin{cases} N \text{ is solvable } \& \\ G/N \text{ is solvable} \end{cases}$

Proof. - (recall, by our very first definition, a group $\mathcal{G}$ is solvable if $\mathcal{G}^{(n)} = \{e\}$ for some $n \geq 0$. see page 1 above.)

($\Rightarrow$). Assume G is solvable and let $n \geq 0$ be so that $G^{(n)} = \{e\}$.

Then $N^{(n)} \subset G^{(n)} = \{e\} \Rightarrow N$ is solvable

(easy induction argument
$N \subset G \checkmark \Rightarrow [N; N] \subset [G; G] \Rightarrow \ldots$)

Note : same argument will also work for any subgroup $H \leq G$ (not nec. normal). Thus $\boxed{\text{Subgroups of solvable groups are solvable}}$.

Now consider $\pi : G \longrightarrow G/N$. We use the fact that π is a group hom. to say $\pi\big([G ; G]\big) = [\pi(G) ; \pi(G)]$ $= [(G/N) ; (G/N)]$. Repeatedly applying this equation gives

$$\underset{}{(G/N)^{(n)}} = \pi\left(G^{(n)}\right) = \pi\Big(\{\underset{\substack{\uparrow \\ \text{unit of } G}}{e_G}\}\Big) = \{\underset{\substack{\uparrow \\ \text{unit of } G/N.}}{e_{G/N}}\}$$

Hence G/N is solvable $\rightarrow$ $\boxed{\text{Quotient gps of Solvable are Solvable}}$.

($\Leftarrow$). Assume N and G/N are solvable. By Theorem (27.2) there exist composition series, with abelian graded pieces :

$$\Sigma_1 : \quad G/N = \overline{G}_0 \trianglerighteq \overline{G}_1 \trianglerighteq \cdots \trianglerighteq \overline{G}_\ell = \{e\}$$

$$\Sigma_2 : \quad N = N_0 \trianglerighteq N_1 \trianglerighteq \cdots \trianglerighteq N_k = \{e\}$$

Again, if $\pi : G \longrightarrow G/N$ is the natural projection then define $\begin{cases} G_j = \pi^{-1}(\overline{G}_j) \subset G & 0 \leq j \leq l \\ G_{l+i} = N_i & 0 \leq i \leq k \end{cases}$

to get

$$\Sigma : G = \underbrace{G_0 \supset G_1 \supset \cdots \supset G_l}_{\downarrow} = N = \underbrace{N_0 \trianglerighteq N_1 \trianglerighteq \cdots \trianglerighteq N_k = \{e\}}_{\text{graded pieces are abelian}}$$

Need to prove (i) $G_j \trianglerighteq G_{j+1}$
(for every $j = 0, 1, \ldots, l-1$.)

(ii) $G_j / G_{j+1} \cong \overline{G}_j / \overline{G}_{j+1}$ (hence abelian)

If we managed to prove (i) and (ii), then by Thm (27.2) again, we will be able to conclude that G is solvable.

<u>Proof of (i) & (ii)</u> :

$$\begin{array}{ccc} G & \xrightarrow{\pi} & \overline{G} = G/N \\ \text{IV} & & \text{IV} \\ G_j & \longrightarrow & \overline{G}_j \end{array}$$

(surjective by defn. of G_j)

denote by π_j

$$\begin{tikzcd} G_j \arrow[r, "\pi_j"] \arrow[rr, bend right, "f_j"'] & \overline{G}_j \arrow[r] & \overline{G}_j / \overline{G}_{j+1} \end{tikzcd}$$

f_j (composition of two gp. homs.)

$\mathrm{Ker}(f_j) = \pi_j^{-1}(\overline{G}_{j+1}) = G_{j+1}$. Hence by 1st iso. thm. (i) & (ii) follow. $\square$