Lecture 28

(28.0) Recall: the summary of the main properties of solvable groups.

(1) $G^{(n)} = \{e\}$ for some $n \geq 0$. Where $G^{(0)} = G$;
$G^{(1)} = [G^{(0)}; G^{(0)}]$; ... (DEFINITION)

(2) There exist a composition series of G with <u>abelian</u> graded pieces. (THM 27.2)

(3) Sub & Quotient groups of G are solvable.

(4) N & G/N solvable $\Rightarrow$ G is solvable

(28.1) If G is finite & solvable, then for a Jordan-Hölder series of G; say $\Sigma : G = H_0 \not\trianglerighteq H_1 \not\trianglerighteq \cdots \not\trianglerighteq H_\ell = \{e\}$; the associated graded pieces are <u>abelian</u> & simple (see Lecture 25, p.2)

Hence $H_j/H_{j+1} \cong \mathbb{Z}/p_j\mathbb{Z}$ ($p_0, p_1, \ldots, p_{\ell-1}$: primes).

Proof. $\Sigma : G = H_0 \not\trianglerighteq H_1 \not\trianglerighteq \cdots \not\trianglerighteq H_\ell = \{e\}$ Jordan-Hölder Series

Let $\Sigma_1 : G = K_0 \trianglerighteq K_1 \trianglerighteq \cdots \trianglerighteq K_s = \{e\}$ be a composition series with <u>abelian</u> <u>graded pieces</u>. Such a composition series exists because G is solvable

Now (Schrier's Thm - Lecture 25 page 2) we can find a common refinement of Σ & Σ_1, say Σ'. Then graded pieces of Σ' are abelian, because this is true for Σ_1 and Σ' refines Σ_1. But graded pieces of Σ' must be those of Σ (& bunch of trivial groups) since Σ is Jordan-Hölder (only admit "silly refinements" -

$$\Sigma' : \quad H_0 = H_0 \cdots = H_0 \underset{\neq}{\trianglerighteq} H_1 = H_1 \cdots = H_1 \underset{\neq}{\trianglerighteq} \cdots \quad)$$

$\square$

(28.2) (Lower) Central Series and Nilpotent groups.

Definition. Let G be a group. Define

$$c^1(G) = G \; ; \quad c^2(G) = [G \,;\, c^1(G)] \quad (\trianglelefteq G)$$

(Lemma 26.1)

$$c^3(G) = [G \,;\, c^2(G)] \quad (\trianglelefteq G)$$

& induction argument.

$$\cdots \quad c^{j+1}(G) = [G \,;\, c^j(G)] \quad (\trianglelefteq G)$$

$$c^1(G) \trianglerighteq c^2(G) \trianglerighteq \cdots \trianglerighteq c^j(G) \trianglerighteq c^{j+1}(G) \trianglerighteq \cdots$$

called <u>central series</u>. (see remark (ii) below - next page.)

We say G is <u>nilpotent</u> if $C^n(G) = \{e\}$ for some $n \geq 1$.

(28.3) Some remarks. (i) $C^2(G) = [G;G] = G^{(1)}$ (from the commutator series)

[Hence, abelian $\Rightarrow$ Nilpotent.]

(ii) $C^{n+1}(G) = [G; C^n(G)]$

is generated by $\{ gxg^{-1}x^{-1} : g \in G,\ x \in C^n(G) \}$.

But $C^n(G) \trianglelefteq G \Rightarrow gxg^{-1} \in C^n(G) \Rightarrow gxg^{-1}x^{-1} \in C^n(G)$.

($\&\ x^{-1} \in C^n(G)$)

This is why we know that $C^{n+1}(G) \trianglelefteq C^n(G)$.

(iii) For a nilpotent group, G, the central series of G is a composition series

$$G = C^1(G) \trianglerighteq C^2(G) \trianglerighteq \dots \trianglerighteq C^m(G) = \{e\}.$$

Moreover, it has the property that

$$C^{n+1}(G) = [G; C^n(G)] \quad (\text{contains } [C^n(G); C^n(G)])$$

Hence $C^n(G) / C^{n+1}(G)$ is abelian. By Theorem (27.2)

we conclude: $\boxed{\text{Nilpotent} \Rightarrow \text{Solvable}}$

(28.4) Let B be the subgroup of $GL_2(\mathbb{R})$ consisting of upper triangular matrices : (see Lecture 26 page 4.)

$$C^1(B) = B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, c \neq 0 \right\}$$

$$\rotatebox[origin=c]{90}{$\trianglerighteq$}$$

$$C^2(B) = [B ; B] = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \text{ arbitrary} \right\}$$

$C^3(B) = [B ; C^2(B)]$. A typical generator of $C^3(B)$ is of the form

$$\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a_1} & \frac{-b_1}{a_1 c_1} \\ 0 & \frac{1}{c_1} \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \quad \text{where } d = \frac{a_1 x - \cancel{b_1}\, c_1 x}{c_1}$$

$$= x \left(\frac{a_1}{c_1} - 1 \right) \text{ not necessarily } 0.$$

By choosing $\frac{a_1}{c_1} \neq 1$, we conclude, $[B ; C^2(B)] = C^2(B)$ $= C^3(B) = \cdots$

Hence B is <u>not</u> nilpotent.

(but B <u>is</u> solvable.)

(28.5) Now we have the following analogue of Theorem 27.2 :

Theorem. - G is nilpotent if and only if it has a composition series $G = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_m = \{e\}$: Σ such that $[G; H_\ell] \subset H_{\ell+1}$ for every $\ell = 0, 1, \ldots, m-1$.

[Hence $[H_\ell; H_\ell] \subset H_{\ell+1} \Rightarrow H_\ell / H_{\ell+1}$ is abelian.]

Proof. ($\Rightarrow$). Just take $H_\ell := C^{\ell+1}(G)$.

($\Leftarrow$). Assume such a composition series Σ exists. We will prove (by induction on ℓ) that $C^{\ell+1}(G) \subset H_\ell$.

Base case: ($\ell = 0$) $\quad C^1(G) = G = H_0$ ✓

Induction step:
$$\begin{aligned} C^{\ell+2}(G) &= [G; C^{\ell+1}(G)] \\ &\subset [G; H_\ell] \quad \text{(induction hypothesis)} \\ &\subset H_{\ell+1} \quad \text{(given)} \end{aligned}$$
□

(28.6) More properties of nilpotent groups, analogous to those of solvable groups (see (28.0) above)

(1) Sub and quotient groups of a nilpotent group are nilpotent. (same idea as in (27.4) - pages 6 & 7 of Lecture 27.)

(2) G is nilpotent if and only if there exists $A \leq Z(G)$ such that G/A is nilpotent.

center of G

Proof. $(\Rightarrow)$. Follows from (1) - just take $A = Z(G)$.

$(\Leftarrow)$. Consider $\pi : G \longrightarrow G/A$. Let $n \geq 1$ be so that $C^n(G/A) = \{e_{G/A}\}$. But then

$$\pi(C^n(G)) = C^n(G/A) = \{e_{G/A}\} \text{ ; meaning}$$

$C^n(G) \subset \mathrm{Ker}(\pi) = A$. But $A \subset Z(G)$, implying

$$C^{n+1}(G) \subseteq [G;A] = \{e_G\} .$$

□

(3) $|G| = p^n \Rightarrow G$ is nilpotent.

(Hint: use (2) above, induction argument & the fact that $Z(G) \neq \{e\}$.)