Lecture 29

(29.0) Recall that we proved the following properties of solvable & nilpotent groups

(1) Nilpotent $\Rightarrow$ Solvable (NOT conversely).

(2) Abelian $\Rightarrow$ Nilpotent

$|G| = p^n$ $\Rightarrow$ Nilpotent

(p: prime)

(3) Sub- & quotient groups of a solvable (resp. nilpotent) group are solvable (resp. nilpotent).

(4) N & G/N solvable $\Rightarrow$ G is solvable.

(5) $A \le Z(G)$, G/A nilpotent $\Rightarrow$ G is nilpotent.

Today we will see what finite nilpotent groups are.

(29.1) Useful lemma. - Let G be a nilpotent group and $H \lneq G$ a proper subgroup. Then $H \subsetneq \underline{N_G}(H)$

normalizer of H, defined as

$\{g \in G$ such that $gHg^{-1} = H\}$

Proof. As G is nilpotent, we have a composition series

$G = K_0 \trianglerighteq K_1 \trianglerighteq \cdots \trianglerighteq K_m = \{e\}$ such that

$$[G; K_\ell] \subset K_{\ell+1} \qquad (\ell = 0, 1, \dots, m-1).$$

Note : For any subgroup $K \leq G$; $[G; K] \subset K$ implies that $K \trianglelefteq G$. This is because $\forall\, g \in G,\ k \in K$, we have

$$g\, k\, g^{-1} k^{-1} \in K \quad \& \quad k \in K \leq G$$
$$\Rightarrow g\, k\, g^{-1} \in K \quad \Rightarrow \quad K \trianglelefteq G.$$

Thus each K_ℓ is normal in G. Let G_ℓ = subgroup of G generated by H and K_ℓ : $G_\ell = K_\ell \cdot H$.

$\boxed{\text{Claim.} \quad G_{\ell+1} \trianglelefteq G_\ell \quad \text{— Proof below —(29.2)}}$

Thus we get

$$G_0 = G \trianglerighteq G_1 \trianglerighteq \quad \dots \quad \trianglerighteq G_m = H \quad \& \quad G \supsetneq H.$$

Let k be such that $(0 \leq k \leq m-1)$

$$G_k \underset{\neq}{\trianglerighteq} G_{k+1} = \dots = G_m = H$$

Then $H \underset{\neq}{\trianglelefteq} G_k$ implies $G_k \subset N_G(H)$, hence

$H \subsetneq N_G(H)$ as we wanted. $\square$

(29.2) Assume G is a group; $N_2 \subset N_1$ two <u>normal</u> subgroups of G such that $[G;N_1] \subset N_2$. Let $H \leq G$.

Then $N_2 \cdot H$ is normal in $N_1 \cdot H$.

Proof.- To prove : $\forall\, a \in N_1 H$, $b \in N_2 H$;

$$a b a^{-1} \in N_2 H.$$

- if $a \in H \subset N_1 \cdot H$ then $a b a^{-1} = \underbrace{(a n_2 a^{-1})}_{\in N_2}\underbrace{(a x a^{-1})}_{\in H}$

  $b = n_2 \cdot x \in N_2 \cdot H$

  because $N_2 \trianglelefteq G$ because $a, x \in H$

$$\Rightarrow a b a^{-1} \in N_2 \cdot H$$

- if $a \in N_1$, $b \in N_2 H$ then $b^{-1} a b a^{-1} \in [G;N_1] \subset N_2$

$$\Rightarrow a b a^{-1} \in b N_2 \subset N_2 \cdot H$$

$\square$

(29.3) "Self-normalizing" lemma for Sylow subgroups.

Let G be a finite group, p : a prime,

$P \in Syl_p(G)$, $N_G(P) \leq L \leq G$.

Then $N_G(L) = L$.

Proof.- $P \leq N_G(P) \leq L$. (all in G & $P \in Syl_p(G)$)

Let $g \in N_G(L)$. Then P and $gPg^{-1} \in Syl_p(L)$.

By Sylow Thm (ii), $\exists\, l \in L$ s.t. $P = l(gPg^{-1})l$.

This means $l \cdot g \in N_G(P) \subset L \Rightarrow g \in L$. $\square$

(29.4) Theorem. The following conditions on a finite group G are equivalent.

(1) G is nilpotent.

(2) Every Sylow p-subgp. is normal.

(3) $G \cong$ a direct product of p-groups

Proof. (1) $\Rightarrow$ (2). Assume G is nilpotent, $P \in Syl_p(G)$.

Let $H = N_G(P)$. If $H = G$, then $P \trianglelefteq G$ and we are done. If $H \subsetneq G$, then

$$\left.\begin{array}{l} \cdot\ H \subsetneq N_G(H) \text{ by } (29.1) \\ \cdot\ N_G(H) = H \text{ by } (29.3) \end{array}\right\} \text{contradiction!}$$

(2) $\Rightarrow$ (3) - exercise.

$(3) \Rightarrow (1)$. Assume $G \cong P_1 \times P_2 \times \dots \times P_\ell$ where

$|P_j| = p_j^{n_j}$ for some primes $p_1, p_2, \dots, p_\ell$ and $n_1, n_2, \dots, n_\ell \in \mathbb{Z}_{\geq 1}$.

We already know that p-groups are nilpotent (Lecture 28 page 6). Thus, we have to prove that direct product of nilpotent groups is nilpotent.

So assume that we have two nilpotent groups G_1 and G_2.

Thus we have 2 composition series

$$\Sigma_1 : \quad G_1 = H_0 \trianglerighteq H_1 \trianglerighteq \dots \trianglerighteq H_s = \{e\}$$

$$\Sigma_2 : \quad G_2 = K_0 \trianglerighteq K_1 \trianglerighteq \dots \trianglerighteq K_t = \{e\}$$

so that $[G_1 ; H_j] \subset H_{j+1}$

$[G_2 ; K_\ell] \subset K_{\ell+1}$.

Take Σ: $G_1 \times G_2 = L_0 \trianglerighteq \underbrace{G_1 \times K_1}_{L_1} \trianglerighteq \underbrace{G_1 \times K_2}_{L_2} \trianglerighteq \dots \trianglerighteq \underbrace{G_1 \times K_t}_{L_t}$

$$L_t = G_1 \times K_t \cong G_1 = H_0$$

$$\{e\} = H_s \trianglelefteq \dots \trianglelefteq H_1 \trianglelefteq G_1 = H_0$$

$$L_{t+s} = H_s, \quad \dots, \quad L_{t+1} = H_1$$

i.e. $L_\ell = G_1 \times K_\ell \trianglelefteq G_1 \times G_2$; $(0 \leq \ell \leq t)$

$$L_{t+j} = H_j \times \{e_2\} \subset G_1 \times G_2$$

It is easy to verify that

$$[G_1 \times G_2 \,;\, L_r] \subset L_{r+1} \qquad (0 \leq r \leq s+t-1).$$

Hence $G_1 \times G_2$ is nilpotent. $\square$

(29.5) In conclusion.

$$\begin{cases} p\text{-groups are nilpotent} \\ \text{Nilpotent} + \text{finite} = \text{direct product of } p\text{-groups} \end{cases}$$