Lecture 30

(30.0) Compute $\mathrm{Aut}_{gp}(D_{2n})$. First as a set, then as a group. Let us assume $n \geq 3$.

$$D_{2n} = \langle s, r \mid s^2 = e,\ r^n = e,\ srs = r^{-1} \rangle$$

$$= \{ e, \underbrace{\boxed{s, sr, \ldots, sr^{n-1}}}_{\text{order} = 2}, \underbrace{r}_{\text{order} = n}, r^2, \ldots, r^{n-1} \}$$

$\phi : D_{2n} \xrightarrow{\sim} D_{2n} \Rightarrow$ $\phi(s)$ has order 2
$\phi(r)$ has order n

so $\phi(r) \in \{ e, r, \ldots, r^{n-1} \}$

$ \phi(r) = r^j$ s.t. $(j, n) = 1$

and $\phi(s) = s.r^i \quad (0 \leq i \leq n-1)$

Will any $i \in \{0, \ldots, n-1\}$, $j \in (\mathbb{Z}/n\mathbb{Z})^\times$ work?

To check: $\phi(s)\phi(r)\phi(s) = \phi(r)^{-1}$.

$$s.r^i.r^j.s\, r^i \overset{?}{=} r^{-j}$$

$$e = s^2\, r^{-i-j+i} \overset{?}{=} r^{-j} \quad \checkmark$$

So $\left|\mathrm{Aut}_{gp}(D_{2n})\right| = n \cdot \underbrace{\varphi(n)}$.

$\varphi(n)$: Euler's φ-fn.

$= \#\{ j \in \{0, \dots, n-1\}$ such that $\gcd(j,n) = 1 \}$

Recall: (Lecture 21)

$$\varphi(n) = \left|(\mathbb{Z}/n\mathbb{Z})^{\times}\right| = \left|\mathrm{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z})\right|$$

so $\mathrm{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z}) \leq \mathrm{Aut}_{gp}(D_{2n})$ — $(*)$

$$\begin{array}{ccc} \mathrm{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z}) & & \mathrm{Aut}_{gp}(D_{2n}) \\ \Psi & & \\ \{1 \mapsto j\} & \longmapsto & \left\{ \begin{array}{l} s \mapsto s \\ r \mapsto r^j \end{array} \right\} =: g_j \end{array}$$

Question: Is $\mathrm{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z})$ normal in $\mathrm{Aut}_{gp}(D_{2n})$?

Take $f : D_{2n} \longrightarrow D_{2n}$. Then

$f(s) = s \cdot r$

$f(r) = r$

- $\underbrace{f \circ f \circ \dots \circ f}_{k\text{-times}} : \begin{array}{l} s \longmapsto s \cdot r^k \\ r \longmapsto r \end{array}$ (so order of f in $\mathrm{Aut}_{gp}(D_{2n})$ is n)

- $f \circ g_j \circ f^{-1} : s \underbrace{\longmapsto}_{f^{-1}} s \cdot r^{-1} \underbrace{\longmapsto}_{g_j} s \cdot r^{-j} \underbrace{\longmapsto}_{f} s r^{1-j}$

$\notin \mathrm{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z})$. so **NO** !

Question 2. Is $\langle f \rangle$ normal in $\mathrm{Aut}_{gp}(D_{2n})$?

Answer. YES.

$g_j \circ f \circ g_j^{-1}$ maps $r \longmapsto r$ [Ex. easy]

$$s \underbrace{\longmapsto}_{g_j^{-1}} s \underset{f}{\longmapsto} s.r \underset{g_j}{\longmapsto} s.r^j$$

$$\Rightarrow \quad g_j \circ f \circ g_j^{-1} = \underbrace{f \circ f \circ \cdots \circ f}_{j\text{-times}}.$$

Conclusion: $\mathrm{Aut}_{gp}(D_{2n}) \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathrm{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z})$.

(30.1) Composition series vs. group hom.

Let $\psi : G \longrightarrow H$ be a group homomorphism.

Assume $H = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_m = \{e\}$

is a composition series of H.

Define $G_j = \psi^{-1}(H_j) \subset G$.

Then (i) $G = G_0 \trianglerighteq G_1 \trianglerighteq \dots \trianglerighteq G_m = \operatorname{Ker}\psi \trianglerighteq \{e\}$

is a composition series.

(ii) $G_j / G_{j+1} \longrightarrow H_j / H_{j+1}$ is an

<u>injective</u> group homomorphism. $\forall\ 0 \le j \le m-1$.

· If we assume ψ is <u>surjective</u> gp. hom. then

$\forall\ 0 \le j \le m-1, \quad G_j / G_{j+1} \cong H_j / H_{j+1}$.

Proof. Uses same arguments as in the 2^{nd} iso. thm.
<u>Lecture 8</u>.

(a) Each $G_j \le G$ because, for any group hom

$f : \mathcal{G} \to \mathcal{G}'$; $f^{-1}(A) \le \mathcal{G}$
$\quad\quad\quad\quad\ \ \ \text{IV}$
$\quad\quad\quad\quad\ \ \ A$

[Need to show : $a, b \in f^{-1}(A)$
$\Rightarrow a^{-1}b \in f^{-1}(A)$ - $\boxed{\text{Lecture 1}}$

i.e. $f(a^{-1}b) \in A$
i.e. $f(a)^{-1} f(b) \in A$ (f is a gp. hom.)
This is true because $f(a), f(b) \in A$
& $A \le \mathcal{G}'$.]

(b) G_{j+1} is normal in G_j because, for any group hom. $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2 \Rightarrow f^{-1}(A)$ is normal in $\mathcal{G}_1$.

$$\begin{array}{c}\mathcal{G}_2 \\ \trianglerighteq \\ A\end{array}$$

[To prove: $g \in \mathcal{G}_1,\ x \in f^{-1}(A) \Rightarrow g x g^{-1} \in f^{-1}(A)$

i.e. $f(g x g^{-1}) \in A$. This is same as saying $f(g) \cdot f(x) \cdot f(g)^{-1} \in A$, which is true because $A \trianglelefteq \mathcal{G}_2$.]

(c) Now ψ restricted to G_j gives a group hom

$$\psi_j : G_j \longrightarrow H_j$$

(need not be surjective unless ψ is surj.)

We compose ψ_j with $H_j \xrightarrow{\pi} H_j / H_{j+1}$.

$$G_j \xrightarrow{\pi \circ \psi_j} H_j / H_{j+1}$$

$$\begin{aligned} \operatorname{Ker}(\pi \circ \psi_j) \ni x \quad &\Leftrightarrow \quad \pi(\psi_j(x)) = e_{H_j/H_{j+1}} \quad \text{(unit)} \\ &\Leftrightarrow \quad \psi_j(x) \in H_{j+1} \\ &\Leftrightarrow \quad x \in \psi^{-1}(H_{j+1}) = G_{j+1} \end{aligned}$$

1st iso. thm. gives an injective gp. hom $\boxed{\text{Lecture 6}}$

$$G_j/G_{j+1} \longrightarrow H_j/H_{j+1}.$$

If ψ was surjective to begin with, $\psi_j : G_j \longrightarrow H_j$ will also be surjective and hence the gp. hom

$G_j/G_{j+1} \longrightarrow H_j/H_{j+1}$ will be both injective & surjective hence an iso. $\square$

(30.2) Jordan-Hölder series.

Let G be a group (NOT necessarily finite) and $N \trianglelefteq G$. Then G admits a Jordan-Hölder series $\Longleftrightarrow$ N & G/N admit a J-H series.

$\ell(G)$, $\ell(N)$, $\ell(G/N)$ = lengths of respective J-H series.

Then $\ell(G) = \ell(N) + \ell(G/N)$.