

# Lecture 30

(30.0) Compute  $\text{Aut}_{gp}(\mathbb{D}_{2n})$ . First as a set, then as a group. Let us assume  $n \geq 3$ .

$$\mathbb{D}_{2n} = \langle s, r \mid s^2 = e, r^n = e, srs = r^{-1} \rangle$$

$$= \left\{ e, \boxed{s, sr, \dots, sr^{n-1}}, \left. \begin{matrix} r, r^2, \dots, r^{n-1} \\ \uparrow \\ \text{order} = n \end{matrix} \right\} \right.$$

$\uparrow$   
order = 2
 $\uparrow$   
order = n

$$\phi : \mathbb{D}_{2n} \xrightarrow{\sim} \mathbb{D}_{2n} \Rightarrow \begin{matrix} \phi(s) \text{ has order } 2 \\ \phi(r) \text{ has order } n \end{matrix}$$

so  $\phi(r) \in \{e, r, \dots, r^{n-1}\}$

$$\parallel$$

$$r^j \text{ s.t. } (j, n) = 1$$

and  $\phi(s) = s \cdot r^i \quad (0 \leq i \leq n-1)$

Will any  $i \in \{0, \dots, n-1\}$ ,  $j \in (\mathbb{Z}/n\mathbb{Z})^\times$  work?

To check:  $\phi(s) \phi(r) \phi(s) = \phi(r)^{-1}$

$$s \cdot r^i \cdot r^j \cdot \boxed{s} \cdot r^i \stackrel{?}{=} r^{-j}$$

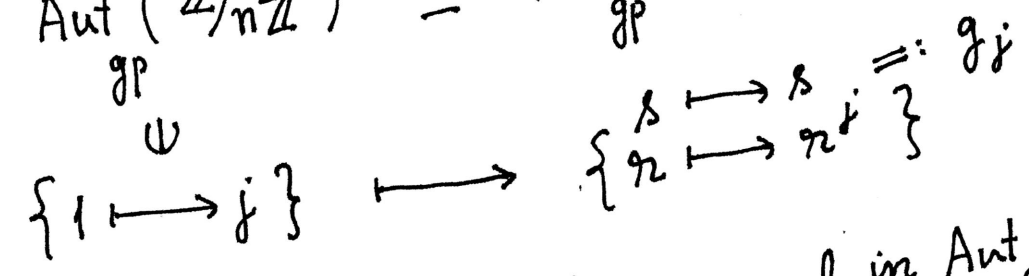
$$e = \boxed{s^2} \cdot r^{-i-j+i} \stackrel{?}{=} r^{-j} \quad \checkmark$$

So  $|\text{Aut}_{gp}(D_{2n})| = n \cdot \underbrace{\varphi(n)}_{\substack{\text{Euler's } \varphi\text{-fn.} \\ = \#\{j \in \{0, \dots, n-1\} \text{ such that} \\ \gcd(j, n) = 1\}}}$

Recall: (Lecture 21)

$\varphi(n) = |(\mathbb{Z}/n\mathbb{Z})^*| = |\text{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z})|$

so  $\text{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z}) \leq \text{Aut}_{gp}(D_{2n}) \quad (*)$



Question: Is  $\text{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z})$  normal in  $\text{Aut}_{gp}(D_{2n})$ ?

Take  $f: D_{2n} \rightarrow D_{2n}$ . Then

$f(s) = s \cdot r$   
 $f(r) = r$

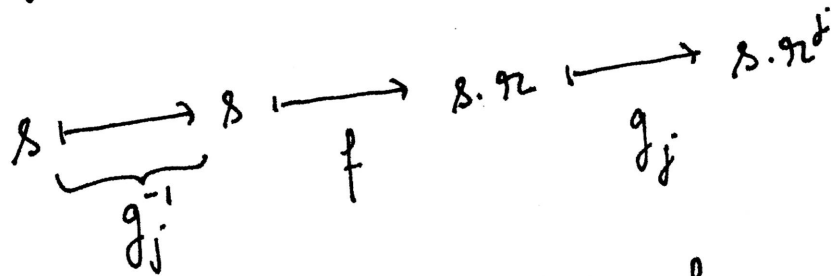
$f \circ f \circ \dots \circ f$  (k-times) :  $\begin{array}{l} s \mapsto s \cdot r^k \\ r \mapsto r \end{array}$  (so order of  $f$  in  $\text{Aut}_{gp}(D_{2n})$  is  $n$ )

$f \circ g_j \circ f^{-1}$  :  $\underbrace{s \mapsto s \cdot r^{-1}}_{f^{-1}} \xrightarrow{g_j} \underbrace{s \cdot r^{-1} \mapsto s \cdot r^{-1} \cdot r^j}_{g_j} \xrightarrow{f} \underbrace{s \cdot r^{-1} \cdot r^j \mapsto s \cdot r^{-1} \cdot r^{j+1}}_f$   
 $\notin \text{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z})$ . so NO!

Question 2. Is  $\langle f \rangle$  normal in  $\text{Aut}_{gp}(D_{2n})$ ? ③

Answer. YES.

$g_j \circ f \circ g_j^{-1}$  maps  $\tau \longmapsto \tau$  [Ex. easy]



$$\Rightarrow g_j \circ f \circ g_j^{-1} = \underbrace{f \circ f \circ \dots \circ f}_{j\text{-times}}$$

Conclusion:  $\text{Aut}_{gp}(D_{2n}) \cong \mathbb{Z}/n\mathbb{Z} \rtimes \text{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z})$ .

(30.1) Composition series vs. group hom.

Let  $\psi: G \longrightarrow H$  be a group homomorphism.

Assume  $H = H_0 \supseteq H_1 \supseteq \dots \supseteq H_m = \{e\}$

is a composition series of  $H$ .

Define  $G_j = \psi^{-1}(H_j) \subset G$ .

Then (i)  $G = G_0 \supseteq G_1 \supseteq \dots \supseteq G_m = \text{Ker } \psi \supseteq \{e\}$  (4)  
 is a composition series.

(ii)  $G_j / G_{j+1} \longrightarrow H_j / H_{j+1}$  is an  
injective group homomorphism.  $\forall 0 \leq j \leq m-1$ .

• If we assume  $\psi$  is surjective gp. hom. then  
 $\forall 0 \leq j \leq m-1, G_j / G_{j+1} \cong H_j / H_{j+1}$ .

Proof. Uses same arguments as in the 2<sup>nd</sup> iso. thm.  
Lecture 8.

(a) Each  $G_j \leq G$  because, for any group hom

$$f: \mathfrak{g} \rightarrow \mathfrak{g}' \quad ; \quad f^{-1}(A) \leq \mathfrak{g}$$

IV  
A

[Need to show:  $a, b \in f^{-1}(A)$   
 $\Rightarrow a^{-1}b \in f^{-1}(A)$  - Lecture 1]

i.e.  $f(a^{-1}b) \in A$

i.e.  $f(a)^{-1}f(b) \in A$  ( $f$  is a gp. hom.)

This is true because  $f(a), f(b) \in A$   
 &  $A \leq \mathfrak{g}'$ .

(b)  $G_{j+1}$  is normal in  $G_j$  because, for any (5)

group hom.  $f: G_1 \longrightarrow G_2 \Rightarrow \bar{f}^{-1}(A)$  is normal in  $G_1$ .

$\downarrow$   
 $A$

[To prove:  $g \in G_1, x \in \bar{f}^{-1}(A) \Rightarrow gxg^{-1} \in \bar{f}^{-1}(A)$   
i.e.  $f(gxg^{-1}) \in A$ . This is same as saying  
 $f(g) \cdot f(x) \cdot f(g)^{-1} \in A$ , which is true because  $A \trianglelefteq G_2$ ]

(c) Now  $\psi$  restricted to  $G_j$  gives a group hom  
 $\psi_j: G_j \longrightarrow H_j$  (need not be surjective  
unless  $\psi$  is surj.)

We compose  $\psi_j$  with  $H_j \xrightarrow{\pi} H_j/H_{j+1}$ .

$$G_j \xrightarrow{\pi \circ \psi_j} H_j/H_{j+1}$$

$$\text{Ker}(\pi \circ \psi_j) \ni x \Leftrightarrow \pi(\psi_j(x)) = e_{H_j/H_{j+1}} \quad \swarrow \text{unit}$$

$$\Leftrightarrow \psi_j(x) \in H_{j+1}$$

$$\Leftrightarrow x \in \psi_j^{-1}(H_{j+1}) = G_{j+1}$$

st

iso. thm. gives an injective gp. hom Lecture 6

$$G_j / G_{j+1} \longrightarrow H_j / H_{j+1}.$$

(6)

If  $\psi$  was surjective to begin with,  $\psi_j : G_j \rightarrow H_j$  will also be surjective and hence the gp. hom

$G_j / G_{j+1} \rightarrow H_j / H_{j+1}$  will be both injective & surjective  
hence an iso.  $\square$

(30.2) Jordan-Hölder series.

Let  $G$  be a group and  $N \trianglelefteq G$ . Then  $G$   
(NOT necessarily finite)

admits a Jordan-Hölder series  $\iff N$  &  $G/N$  admit  
a J-H series.

$l(G), l(N), l(G/N) =$  lengths of respective  
J-H series.

Then  $l(G) = l(N) + l(G/N)$ .