# Lecture 34

(34.0) Recall - an ideal (left) $I$ of a ring $R$ is an abelian subgroup $I \leq R$ such that $RI \subset I$.

$$RI = \{r.x \mid r \in R,\ x \in I\}$$

Some operations on ideals (left).

(1) Intersection. We can take intersection of any set of ideals.

Let $I_\alpha \subset R$ be an ideal (left) for $\alpha$ labelling elements of a set (indexing set) $A$

$$I := \bigcap_{\alpha \in A} I_\alpha \quad \left( = \{a \in R \mid a \in I_\alpha \ \forall \alpha \in A\} \right)$$

Then $I \subset R$ is again an ideal (left).

Pf. $I$ is clearly a subgroup. Now $\forall\, r \in R,\ x \in I$, we have

$r.x \in I_\alpha \ (\forall\, \alpha \in A) \Rightarrow r.x \in I$.

(because $x \in I_\alpha \ \forall \alpha$)

(Proof, in any case)

$0 \in I_\alpha \ \forall \alpha \Rightarrow 0 \in I$

$a, b \in I \Rightarrow a, b \in I_\alpha \ (\forall \alpha) \Rightarrow a \pm b \in I_\alpha \ (\forall \alpha)$

$\Rightarrow a \pm b \in I$ $\square$

(2) Sum of ideals. Again, if $I_\alpha \subset R$ is an ideal (left) $\forall \alpha \in A$ (some set)

$$I = \sum_{\alpha \in A} I_\alpha \quad \Big( = \text{smallest } \underset{(\text{left})}{\text{ideal}} \text{ of } R \text{ containing } \underline{\text{all } I_\alpha\text{'s.}} \Big)$$

↖ just a notation for now

<u>Claim.</u> $I = \left\{ a_{\alpha_1} + a_{\alpha_2} + \dots + a_{\alpha_N} \;\middle|\; \begin{array}{l} N \in \mathbb{Z}_{\geq 0} \\ a_{\alpha_i} \in I_{\alpha_i} \\ \alpha_1, \dots, \alpha_N \in A \end{array} \right\}$

(so $I$ consists of all <u>finite</u> <u>sums</u> we can form using elements of $I_\alpha$'s.)

e.g. If we are only dealing with finitely many $\underset{(\text{left})}{\text{ideals}}$, say, $I_1, \dots, I_n \subset R$; any $\underset{(\text{left})}{\text{ideal}}$ $\tilde{I} \subset R$ containing all $I_1, \dots, I_n$ must contain the set

$$I_1 + \dots + I_n \overset{\text{defn}}{:=} \{ a_1 + \dots + a_n \mid a_1 \in I_1 ; a_2 \in I_2 ; \dots ; a_n \in I_n \}$$

But this set itself is an $\underset{(\text{left})}{\text{ideal}}$ (easy exercise).

Hence our claim (for finite case. The general case follows the exact same argument). $\square$

(34.1) (Left) ideals generated by subsets.

Let $R$ be a ring and $X \subset R$ a subset. Let

$I_X :=$ smallest (left) ideal (generated by) containing the set $X$.
(denoted by $(X)_L$ ← for left … for now at least).

Following operation of intersection, we get

$$I_X = \bigcap_{\substack{\tilde{I} \subset R \text{ is a (left) ideal; and} \\ X \subset \tilde{I}}} \tilde{I}$$

(intersection of (left) ideals is a (left) ideal)

Following the operation of sum, we get

$$I_X = \sum_{x \in X} \boxed{R \cdot x}$$

↑ left ideal containing $x$.

i.e. $I_X$ consist of all finite sums

$$\left\{ r_1 x_1 + r_2 x_2 + \dots + r_N x_N \;\middle|\; \begin{array}{l} r_1, \dots, r_N \in R \\ x_1, \dots, x_N \in X \end{array} \right\}$$

Notational Convenience: if $X = \{x_1, \dots, x_n\}$ (finite subset of $R$)

we just write $(x_1, x_2, \dots, x_n)_L$ for (left) ideal generated by $X$.

optional for comm. rings

(34.2) Let $R$ be a commutative ring and $I \subset R$ be an ideal. We say $I$ is a <u>principal ideal</u>, if $I = (a)$ for some $a \in I$.

e.g. Set of ideals of $\mathbb{Z}$ $\leftrightarrow$ $\{ n\mathbb{Z} : n = 0, 1, 2, \dots \}$

$(n)$ hence <u>principal</u>.

So we have proved (earlier) : $\boxed{\text{every ideal in } \mathbb{Z} \text{ is principal}}$.

(34.3) Product of ideals. Again $R$ is a commutative ring.

$I_1, I_2 \subset R$ ideals in $R$.

<u>Definition.</u> $I_1 \cdot I_2$ is defined as the ideal of $R$ consisting of elements $a_1 b_1 + a_2 b_2 + \dots + a_N b_N$ where

$a_1, \dots, a_N \in I_1$

$b_1, \dots, b_N \in I_2$.

In other words: $I_1 \cdot I_2$ is the ideal generated by the set, say, $I_1 * I_2 = \{ ab \mid a \in I_1, b \in I_2 \}$.

<u>Remark.</u> (1) If $I_1 = (x)$ is a principal ideal, then $I_1 * I_2 = I_1 \cdot I_2$

(Pf. $I_1 = (x)$ ; $I_2$ arbitrary ideal.

$$I_1 * I_2 = \{ rxy \mid y \in I_2, r \in R \}$$

$I_1 * I_2$

is already an ideal : $r_1 x y_1 + r_2 x y_2 = x \underbrace{(r_1 y_1 + r_2 y_2)}_{\text{new } y}$ $\in I_1 * I_2$

$$r'(rxy) = r'r\, xy \in I_1 * I_2 .)$$

(2) Let $R = \mathbb{C}[x,y]$ polynomials in 2 variables.

$$\begin{array}{c} R \\ \cup \\ I \end{array} = (x,y) = \{ f(x,y) \mid f(0,0) = 0 \}$$

$I * I \ni y^2, x^3$ but (check this) not $y^2 - x^3$.

(34.4) Again let $R$ be a commutative ring and $I_1, I_2$ be two ideals in $R$. We say $I_1$ and $I_2$ are coprime if $I_1 + I_2 = R$.

Lemma. If $I_1$ and $I_2$ are coprime, then $I_1 \cdot I_2 = I_1 \cap I_2$.

In general, we always have $I_1 \cdot I_2 \subset I_1 \cap I_2$.

Proof. $I_1 \cdot I_2$ is generated by $a \cdot b$ where $a \in I_1$, $b \in I_2$.

Now $ab \in I_1$ (because $a$ does) $\Rightarrow ab \in I_1 \cap I_2$.
$ab \in I_2$ (because $b$ does)

As $I_1 \cdot I_2$ is generated by such products, we get $I_1 \cdot I_2 \subset I_1 \cap I_2$ ∎

If $I_1 + I_2 = R$, we can find $a \in I_1$ s.t. $1-a \in I_2$.

Hence, $\forall\ x \in I_1 \cap I_2$, we have

$$x = x\cdot(a+1-a) = \underset{\substack{\cap \\ I_2\cdot I_1}}{\boxed{x\cdot a}} + \underset{\substack{\cap \\ I_1\cdot I_2}}{\boxed{x\cdot(1-a)}} \in I_1\cdot I_2.$$

□

(34.5) Lemma. –
$$\left.\begin{array}{l} I_1 + J = R \\ I_2 + J = R \end{array}\right\} \Rightarrow I_1\cdot I_2 + J = R.$$

comm. ring

( $I_1, I_2, J \subset R$ are ideals. $(I_1, J)$, $(I_2, J)$ are coprime.

Then $(I_1\cdot I_2, J)$ are coprime as well. )

Pf

$x_1 + y_1 = 1 \quad (x_1 \in I_1;\ y_1 \in J_1)$.

$x_2 + y_2 = 1 \quad (x_2 \in I_2;\ y_2 \in J)$.

$$\Rightarrow 1 = (x_1+y_1)(x_2+y_2) = \underset{\substack{\cap \\ I_1\cdot I_2}}{x_1 x_2} + \underset{\substack{\cap \\ J}}{\boxed{x_1y_2 + y_1x_2 + y_1y_2}}$$

□

(34.6) Again, let $R$ be a commutative ring and $I_1, I_2 \subset R$ be two ideals, which are <u>coprime</u>.

<u>Thm.</u>
$$R/I_1\cdot I_2 \cong R/I_1 \times R/I_2$$

Direct product of rings: $R_1, R_2$ : rings.

$$R_1 \times R_2 = \{ (a_1, a_2) : a_1 \in R_1;\ a_2 \in R_2 \}$$

(Cartesian product as a set)

Addition & Multiplication: Component-wise.

$(0,0)$ = additively neutral

$(1,1)$ = multiplicatively neutral.

Proof of Thm. Consider the ring hom

$$\begin{array}{ccc} R & \xrightarrow{f} & R/I_1 \times R/I_2 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ a & \longmapsto & (a \pmod{I_1},\ a \pmod{I_2}) \end{array}$$

$$\mathrm{Ker}(f) = I_1 \cap I_2 = I_1 \cdot I_2 \quad \text{(by coprimality)}$$

Choose $a \in I_1$ such that $1 - a \in I_2$
$b \in I_2$ such that $1 - b \in I_1$

(coprime: $I_1 + I_2 = R$)

Then $f(a) = (0,1)$
$f(b) = (1,0)$ $\Rightarrow f(ax + by) = (x, y) \in \mathrm{Im}(f)$

So $f$ is surjective. By $1^{st}$ iso thm $R/\mathrm{Ker} f \cong \mathrm{Im}(f)$

$$\Rightarrow R/_{I_1 \cdot I_2} \cong R/I_1 \times R/I_2. \qquad \square$$

(34.7) Exercise: Carry out the induction step to prove (using Lemmas (34.4) & (34.5) ; Thm (34.6))

- If $I_1, \dots, I_\ell \subset R$ are ideals which are pairwise coprime (i.e. $I_i + I_j = R \quad \forall\, i \neq j$) ; then

$$I_1 \cdot \dots \cdot I_\ell = I_1 \cap \dots \cap I_\ell$$

$$R/I_1 \cdot \dots \cdot I_\ell \cong R/I_1 \times R/I_2 \times \dots \times R/I_\ell .$$