Lecture 37

(37.0) Let R be a commutative ring. Recall that a proper ideal $I \subsetneq R$ is called a prime ideal iff R/I is an integral domain. This is further equivalent to $\boxed{ab \in I \Rightarrow a \in I \text{ or } b \in I}$

Similarly, we say I is maximal iff R/I is a field. Equivalently, I is maximal among all proper ideals of R, with respect to inclusion (i.e. $I \subset \underbrace{J}_{\text{another ideal}} \subset R \Rightarrow I = J$ or $J = R$.)

(37.1) Remark. We have not yet established the existence of maximal ideals in a commutative ring R. In general, this is achieved by using Zorn's lemma, and we will see this proof below. For a particular class of rings, called Noetherian rings, an alternate proof can be given (later in the course) which avoids the use of Zorn's lemma.

(37.2) Geometric look at commutative rings. [Optional - but recommended!]

Geometrically, we can think of commutative rings as rings of functions valued in a field (say $\mathbb{C}$ to fix ideas).

Heuristically

X is space $\rightsquigarrow$ $\mathrm{Fun}(X;\mathbb{C}) = \{f : X \to \mathbb{C}\}$

[adjective - topological] [adjective: continuous]

[algebraic adjectives: polynomial / rational functions.

$Y \subset X \rightsquigarrow I_Y = \{f : X \to \mathbb{C} \mid f(y) = 0 \ \forall y \in Y\}$

"ideals correspond to (closed) subsets"

e.g. $X = \mathbb{R}^2 \rightsquigarrow R =$ polynomial "(real-valued) functions on X

$= \mathbb{R}[x,y]$

Q. What do we gain? A. (1) Can detect non-transversal intersections (i.e. multiplicities).

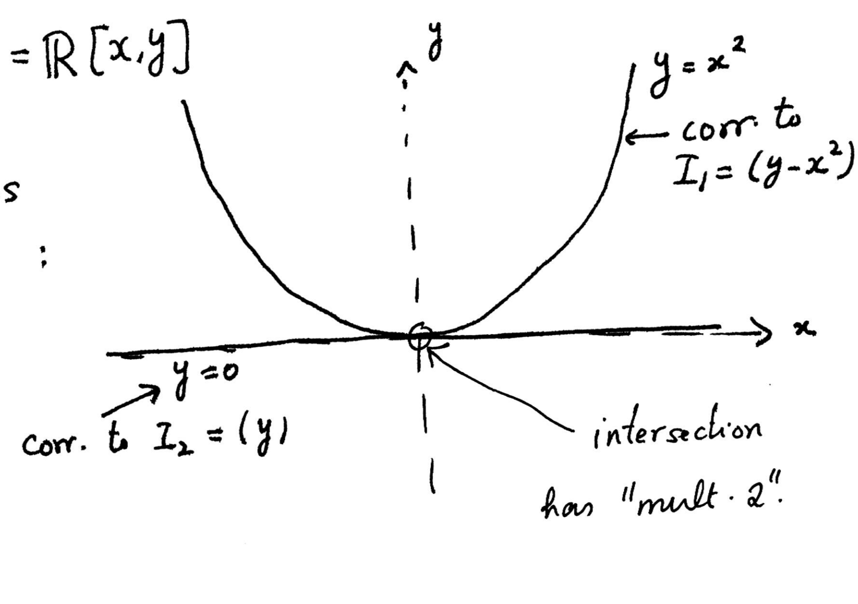
e.g. $I_1 = (y - x^2) \subset R = \mathbb{R}[x,y]$

$I_2 = (y) \subset R = \mathbb{R}[x,y]$

Subsets of $\mathbb{R}^2$ where functions from I_1 (resp. I_2) vanish :

$X_1 = \{(a,b) \in \mathbb{R}^2 \mid b = a^2\}$

$X_2 = \{(a,0) \in \mathbb{R}^2 \mid a \in \mathbb{R}\}$



Intersection of sets $X_1 \cap X_2 = \{(0,0)\}$ but we seem to have lost the "multiplicity".

Right idea :- define the ideal of functions vanishing at $X_1 \cap X_2$.

$$I = I_1 + I_2 = (y - x^2, y) = (y, x^2) \quad (\underline{\text{NOT}}\ (y, x))$$

we get the multiplicity we wanted.

(37.3) Proposition. – Let R be a commutative ring. $J \subsetneq R$ a proper ideal. Then $\exists\, M \subsetneq R$ a maximal ideal s.t. $J \subset M$.

Proof. As remarked earlier, this proof uses Zorn's lemma.

<u>Zorn's Lemma.</u> Let $\mathbb{I} \neq \phi$ be a non-empty set. $\leq$: a partial order on $\mathbb{I}$. $\Big\}$ setup of Zorn's Lemma.

[Meaning : $i \leq i \ (\forall\, i \in \mathbb{I})$ $\quad i \leq j \ \&\ j \leq k \Rightarrow i \leq k \ \forall\, i, j, k \in \mathbb{I}$

$i \leq j \ \&\ j \leq i \Rightarrow i = j \ \forall\, i, j \in \mathbb{I}$.

"Partial" means - given $i, j \in \mathbb{I}$, <u>it is possible that neither $i \leq j$ nor $j \leq i$ hold.</u>

$\hookrightarrow$ i.e., not every pair of elements of $\mathbb{I}$ are comparable.]

<u>Hypothesis</u>: Given $i_0 \leq i_1 \leq i_2 \leq \ldots$ in $\mathbb{I}$
(of Zorn's Lemma)
we can find $j \in \mathbb{I}$ s.t. $i_0 \leq j$; $i_1 \leq j$; $\ldots$

[usually stated as: every <u>chain</u> in $\mathbb{I}$ has a maximal element.]

<u>Conclusion</u>: There exist maximal elements in $\mathbb{I}$.
(of Zorn's Lemma)

In our case: $\mathbb{I}$ = set of proper ideals of R which contain J.

($\mathbb{I} \neq \phi$ because $J \in \mathbb{I}$)

$\leq$ = inclusion ($I_1, I_2 \in \mathbb{I}$, $I_1 \leq I_2$ means $I_1 \subset I_2$)

Verify the hypothesis of Zorn's lemma:

Assume we are given a chain in $\mathbb{I}$

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$$

i.e. each $I_k \subsetneq R$ is a <u>proper ideal</u>, containing J, and $I_k \subset I_{k+1}$ $\forall\, k = 0, 1, 2, \ldots$

Take $I = \bigcup_{k=0}^{\infty} I_k$.

To prove: $I \subsetneq R$ is a <u>proper</u> ideal (because, if $1 \in I$, then $1 \in I_n$ for some $n \geq 0$ $\Rightarrow I_n = I_{n+1} = \ldots = R$ contradicts the fact that each $I_k \subsetneq R$)

$I \supset J$ (clear) ✓

<u>Why is I an ideal?</u>

$a, b \in I \Rightarrow \exists\ n \geq 0$ s.t. $a, b \in I_n$ (hence $a, b \in I_{n+\ell}$ $\forall \ell \geq 0$)

$$\Rightarrow \begin{matrix} a \pm b \in I_n \\ r.a \in I_n \\ (\forall r \in R) \end{matrix} \Rightarrow \begin{matrix} a \pm b \in I \\ ra \in I \\ (\forall r \in R) \end{matrix}.$$ Hence I is an ideal.

We have, therefore, found $I \in \mathbb{I}$ which is greater than all I_k's $(k = 0, 1, \dots)$.

Thus, Zorn's Lemma applies and we have maximal ideals. $\square$

(37.4) Proposition. (1) Any two distinct maximal ideals in a commutative ring R are coprime.

(2) Let $f : R_1 \longrightarrow R_2$ be a ring hom. between two commutative rings; $P_2 \subsetneq R_2$ be a prime ideal. Then $P_1 = f^{-1}(P_2) := \{ a \in R_1 \mid f(a) \in P_2 \}$ is again a prime ideal (in R_1).

Proof. (1) If $M_1 \subsetneq R$ and $M_2 \subsetneq R$ are maximal ideals then $M = M_1 + M_2$ contains both M_1 & M_2.

By maximality (say, of M_1) $M = M_1$ (i.e. $M_2 \subset M_1 \Rightarrow M_1 = M_2$ by max. of M_2)

or $M = R$ (i.e. M_1 & M_2 are coprime).

(2): Consider the ring hom.

$$\begin{array}{ccc} R_1 & \xrightarrow{\bar{f}} & R_2/P_2 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ a & \longmapsto & f(a) \pmod{P_2} \end{array}$$

$\bar{f}$ is a ring hom. because it is a composition of

$$R_1 \xrightarrow{f} R_2 \xrightarrow{\pi} R_2/P_2$$

$\mathrm{Ker}(\bar{f}) = P_1 = \{a \in R_1 \mid f(a) \in P_2\}$. So by 1^{st} iso. thm. we get an <u>injective</u> ring hom $R_1/P_1 \overset{i}{\hookrightarrow} R_2/P_2$

this tiny hook is meant to indicate <u>injective</u>.

But R_2/P_2 is an integral domain (because $P_2 \subsetneq R_2$ is prime)

and, hence, so is R_1/P_1

(because $a.b = 0$ in R_1/P_1 $\Rightarrow$ $i(a)\, i(b) = 0$ in R_2/P_2

$\Rightarrow$ $i(a) = 0$ or $i(b) = 0$ in R_2/P_2 (as it is a domain) $\Rightarrow$ $a = 0$ or $b = 0$ because i is injective