Lecture 38

(38.0) Recall that we proved in Lecture 37:

(1) Maximal ideals exist.

(2) $M_1, M_2 \subsetneq R$ maximal ideals $\Rightarrow \begin{cases} M_1 = M_2 \text{ ; or} \\ M_1 + M_2 = R \end{cases}$

(3) For a ring hom. $f : R_1 \longrightarrow R_2$, $P_2 \subsetneq R_2$ prime ideal in R_2,

the set $P_1 = \{ a \in R_1 \mid f(a) \in P_2 \} \subsetneq R_1$ is a prime ideal.

<u>Remark.</u> The conclusion of (3) will be false if we replaced the adjective "prime" by maximal

e.g. $f : \mathbb{Z} \longrightarrow \mathbb{Q}$, $n \longmapsto \frac{n}{1}$

$(0) \subsetneq \mathbb{Q}$ is (the only) maximal ideal

But $(0) \subsetneq \mathbb{Z}$ is not a maximal ideal (still prime ideal!)

$(0) = \{ n \mid f(n) = 0 \}$

(38.1) Local ring. – Let R be a commutative ring. We say that R is <u>local</u> if it has <u>a unique</u> maximal ideal.

e.g. every field is local. ($(0) \subsetneq K$ is the only <u>proper</u> ideal).

<u>Prop.</u> A commutative ring R is local if, and only if

$M = R \setminus R^\times = \{ a \in R \mid a \text{ is } \underline{\text{NOT}} \text{ a unit} \} \subsetneq R$ is an ideal.

$R^\times = \{ x \in R \mid xy = 1 \text{ for some } y \in R \}$

In this case, $M \subsetneq R$ is the unique maximal ideal of R.

Proof. - We begin by recalling that $I \subsetneq R$ proper ideal $\Rightarrow$ $I \cap R^\times = \phi$. That is, $I \subset M$. Thus $M \subsetneq R$, <u>if an ideal</u>, is clearly <u>the</u> maximal one.

If $M \subsetneq R$ is an ideal, then by the observation made above,

Set of maximal ideals of R = $\{M\}$; i.e. R is <u>local</u>.

Conversely, if R is local and $J \subsetneq R$ is its unique maximal ideal; then $J \subset M$ (after all J is an ideal!). And if $x \in M$

$I := (x) \subsetneq R$, by our previous result, $I \subset \underbrace{\tilde{I}}_{\text{a max'l ideal}} \subsetneq R$.

(proper because $I = \{rx \mid r \in R\}$ and $1 \in I \Leftrightarrow x \in R^\times$ i.e. $I \subsetneq R \Leftrightarrow x \in M = R \setminus R^\times$)

As J is the only max'l ideal, $\tilde{I} = J$ and hence $x \in J$.

Thus $J = M$ (is an ideal) $\square$

(38.2) Examples. $R = \underbrace{K}_{\text{a field}}[x]/(x^2)$ is local.

Proof. Let us figure out $R^\times$. $a + bx \in R^\times$ $(a, b \in K)$

$\Leftrightarrow$ $(a+bx)(c+dx) = 1$ for some $c, d \in K$

i.e. $ac + (ad+bc)x = 1$ $\equiv$ $ac = 1$ & $ad + bc = 0$

can be solved $\quad c = \frac{1}{a}$, $\quad d = -\frac{bc}{a}$ $\quad \Longleftrightarrow \quad a \in K^\times = K \setminus \{0\}$

Thus $\quad R^\times = \left\{ a + bx \;\middle|\; \begin{array}{l} a \neq 0 \\ (a, b \in K) \end{array} \right\} \subset R$

$\Rightarrow \quad R \setminus R^\times = (x)$ is an ideal. Hence R is local &

$(x) \subsetneq R$ is its unique maximal ideal. Hence:

Set of ideals of $\underset{R =}{K[x]/(x^2)} = \{ (0) \,;\, (x) \,;\, \underset{R}{\overset{\parallel}{(1)}} \}$

only prime/maximal

(because $x^2 = 0$)

(38.3) Example. $\quad R = K[\![x]\!]$ ring of formal series in 1 variable with coefficients from a field K.

(instead of polynomials $\rightsquigarrow K[x]$)

A typical element of R is of the form

$$a_0 + a_1 x + a_2 x^2 + \ldots \text{ (∞ly many terms)} \underset{\text{(NOTATION)}}{=} \sum_{j=0}^{\infty} a_j x^j$$

Addition: (componentwise) $\quad \left(\sum_{j=0}^{\infty} a_j x^j \right) + \left(\sum_{j=0}^{\infty} a_j' x^j \right) = \sum_{j=0}^{\infty} (a_j + a_j') x^j$

Multiplication. (distribute over +)

$$\left(\sum_{j=0}^{\infty} a_j x^j \right) \left(\sum_{k=0}^{\infty} b_k x^k \right) = \sum_{N=0}^{\infty} \left(\underline{\sum_{j=0}^{N} a_j b_{N-j}} \right) x^N$$

Note: in our definition of $+$ and $\cdot$ of elements of $R = K[[x]]$ only finitely many operations were performed to get the coeff. of x^n (fixed n).

see e.g. $\sum_{j=0}^{N} a_j \cdot b_{N-j} \in K$ for $\begin{matrix} a_0 \cdots a_N \\ b_0 \cdots b_N \end{matrix} \in K$.

It will give us a ring structure on, say for example, ring of Laurent Series $\left\{ \sum_{j=-N}^{\infty} a_j \cdot x^j \; : \; a_i \in K \; (i = -N, -N+1, \dots) \right\}$

some finite $N \in \mathbb{Z}_{\geq 0}$

Sometimes denoted by $K((x))$; sometimes by $K[x^{-1}; x]]$

Fun exercise. Our definition (of mult.) will not make sense for the abelian group $\left\{ \sum_{j=-\infty}^{\infty} a_j \cdot x^j \; : \; a_j \in K \; \forall j \in \mathbb{Z} \right\} = K[[x^{-1}, x]]$

(Because, if it did,

$$\left.\begin{aligned} &\dots + x^{-2} + x^{-1} + \underbrace{1 + x + x^2 + \dots}_{} \\ &= \frac{x^{-1}}{1 - x^{-1}} + \frac{1}{1-x} = 0 \end{aligned}\right\}$$

compare coeff of x^k to get $1 = 0$!)

Same calculation as in Example (38.2) gives:

$R = K[[x]]$; $R^{\times} = K^{\times} + x K[[x]]$

$\Longrightarrow$ $R \setminus R^{\times} = (x)$ is an ideal.

Hence $(R, (x))$ is a local ring.

Notation: "(R, M) is a local ring" $\rightsquigarrow M \subsetneq R$ is the unique maximal ideal of R.

<u>Example (38.4)</u> $R = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; \begin{array}{l} \gcd(a,b) = 1, \ \& \\ p \text{ does not divide } b \\ (\text{i.e. } b \notin p\cdot\mathbb{Z}) \end{array} \right\}$

($p \in \mathbb{Z}_{\geq 2}$ fixed prime)

<u>R is a ring (subring of $\mathbb{Q}$)</u> $\quad \frac{0}{1}, \frac{1}{1} \in R$ ✓

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{a_1 b_2 + a_2 b_1}{b_1 b_2}$$

$p \nmid b_1$ & $p \nmid b_2$ $\Longrightarrow p \nmid b_1 b_2$ (read: does not divide)

$\Longrightarrow p$ does not divide any factor of $b_1 b_2$ $\Longrightarrow \frac{a_1}{b_1} + \frac{a_2}{b_2} \in R$

& $\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \in R$

$$R^\times = \left\{ \frac{a}{b} \;\middle|\; \begin{array}{c} a, b \in \mathbb{Z}_{\neq 0} \\ \gcd(a,b) = 1 \\ p \text{ does not divide } a \text{ or } b \end{array} \right\} \subset R$$

$\Longrightarrow R \setminus R^\times = (p) = p\cdot R$ is an ideal.

(38.5) [Optional but recommended]

Geometric viewpoint towards commutative rings.

Commutative ring R = [Type] functions on [Type] Space X valued in $\mathbb{C}$ (say) or some field.

For instance : continuous — topological

polynomial — vector

⋮ — ⋮

Ideals = subset of functions which vanish on a given subset $Y \subset X$ (in topolical setting- it must be a closed set)

"open sets" must then be given by non-vanishing e.g.

$$\left(\text{e.g. } GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \;\middle|\; ad - bc \neq 0 \right\} \right)$$

right adjective in Ring column = Multiplicatively closed set.

(38.6) Definition. Let R be a commutative ring. $S \subset R$ is said to be multiplicatively closed if

(i) $0 \notin S$ (ii) $1 \in S$

(iii) $a, b \in S \Rightarrow ab \in S$.