Lecture 39

(39.0) Recall - we defined a local ring as a (commutative) ring R that has a unique maximal ideal $M \subsetneq R$. We proved that R is local $\iff$ $R \setminus R^\times$ is an ideal (= the unique max'l ideal).

Our main examples of local rings:

$K[x]/(x^2)$; $K[[x]]$; $\mathbb{Z}_p$

Quotient ✓ [Already discussed]

Completion [Later]

Localization (Special case of forming ring of fractions) [TODAY's topic] ← Operations on $K[x]$, $\mathbb{Z}$ to get these rings

(39.1) Let R be a commutative ring and $S \subset R$. We say that S is <u>multiplicatively closed</u> if

$0 \notin S$ and $a, b \in S \Rightarrow a \cdot b \in S$.

$1 \in S$

We are going to define a new ring - denoted by $S^{-1}R$ - "by inverting elements of S". [Keep in mind: How $\mathbb{Q}$ is defined starting from $\mathbb{Z}$ and inverting every non-zero $n \in \mathbb{Z}_{\neq 0}$.]

playing role of S

(39.2) As a set $S^{-1}R = \dfrac{S \times R}{\text{an equivalence relation } '\sim'}$

Definition. Given $(s_1, r_1) \in S \times R$ (cartesian product of sets)
$(s_2, r_2) \in S \times R$

We say $(s_1, r_1) \sim (s_2, r_2)$ if there exists $t \in S$ such that

$$\boxed{t(s_2 r_1 - s_1 r_2) = 0} \quad \text{——} \quad (*)$$

Remark. In our textbook, it is further assumed that a mult. closed set does not contain zero-divisors. With this stronger hypothesis we don't need to include 't' in the definition above.

We will not make this stronger assumption. For us, a mult. closed set is as defined in (39.1) above.

Lemma. – The relation $\sim$ defined above is an equivalence relation. Meaning: (i) $(s, r) \sim (s, r)$ $\forall\, s \in S,\ r \in R$.

(ii) $(s_1, r_1) \sim (s_2, r_2) \Rightarrow (s_2, r_2) \sim (s_1, r_1)$
$\forall\, s_1, s_2 \in S;\ r_1, r_2 \in R$.

(iii) $(s_1, r_1) \sim (s_2, r_2)$ & $(s_2, r_2) \sim (s_3, r_3)$
$\Rightarrow (s_1, r_1) \sim (s_3, r_3)$

Proof. – (i) is true because, with $t = 1 \in S$, in $(*)$

$$1(sr - sr) = 0$$

(ii) We are given $(s_1, r_1) \sim (s_2, r_2)$; meaning we are given $t \in S$ s.t $t(s_2 r_1 - s_1 r_2) = 0$

But this implies $t(s_1 r_2 - s_2 r_1) = 0$, i.e. $(s_2, r_2) \sim (s_1, r_1)$

(iii) We are given t and t' in S so that

$t(s_1 r_2 - s_2 r_1) = 0$ and $\boxed{t'(s_2 r_3 - s_3 r_2) = 0}$

[i.e. $(s_1, r_1) \sim (s_2, r_2)$ and $(s_2, r_2) \sim (s_3, r_3)$.]

<u>To find</u>: $t'' \in S$ such that $t''(s_1 r_3 - s_3 r_1) = 0$.

<u>Answer</u>: $t'' = \boxed{t \cdot t' \cdot s_2} \in S$.

Check $\boxed{t t' s_2}(s_1 r_3) = t s_1 (s_2 t' r_3)$

$= t s_1 (t' s_3 r_2)$ using

$= t' s_3 (t s_1 r_2) = t' s_3 (t s_2 r_1)$ (using the 1st one)

$= \boxed{t t' s_2}(s_3 r_1)$ as desired. □

(39.3) <u>Notation</u>. A typical element of $S^{-1}R$ is <u>an equivalence class</u> in $S \times R$ <u>modulo</u> $\sim$.

chosen

$\dfrac{r}{s} = [(s, r)]$ (eq.-class mod $\sim$) $\left(\text{i.e.} = \left\{ \begin{array}{l} (s', r') \\ \in S \times R \end{array} \middle| (s', r') \sim (s, r) \right\} \right)$

So $\boxed{\dfrac{r}{s} = \dfrac{0}{1} \iff \exists\, t \in S \text{ such that } t \cdot r = 0}$

(39.4) Ring Structure – Addition and multiplication on $S^{-1}R$:- (usual formulae - copied from $\mathbb{Q}$)

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} := \frac{s_2 r_1 + s_1 r_2}{s_1 s_2}$$

We still have to make sure these are well-defined.

$$\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} := \frac{r_1 r_2}{s_1 s_2}$$

$\frac{0}{1} \rightsquigarrow$ additively neutral $\qquad \frac{1}{1} \rightsquigarrow$ mult. neutral

Prop. (1) These operations are well-defined.

(2) $\left(S^{-1}R, +, \frac{0}{1}\right)$ is an abelian group.

(3) $\cdot$ is associative ; $\frac{1}{1}$ is mult. neutral ($\neq \frac{0}{1}$)

and multiplication distributes over addition.

Proof. We only prove (1). (2) & (3) are (really) easy from the explicit formulae. (e.g. $\frac{1}{1} = \frac{0}{1} \iff \exists\, t \in S$ so that $t.1 = 0$

i.e. $0 \in S$

As $0 \notin S$ (by our defn.) ; $\frac{1}{1} \neq \frac{0}{1}$ in $S^{-1}R$.)

• Addition is well-defined.

We are given $\frac{r_1}{s_1} = \frac{r_1'}{s_1'}$. Let us first prove that

and $\frac{r_2}{s_2}$

$$\frac{s_2 r_1 + s_1 r_2}{s_1 s_2} = \frac{s_2 r_1' + s_1' r_2}{s_1' s_2}$$

[You can repeat the same argument one more time to prove (as we did in class)

$$\frac{r_1}{s_1} = \frac{r_1'}{s_1'} \quad \& \quad \frac{r_2}{s_2} = \frac{r_2'}{s_2'} \Rightarrow \frac{s_2 r_1 + s_1 r_2}{s_1 s_2} = \frac{s_2' r_1' + s_1' r_2'}{s_1' s_2'} \,. \;]$$

Meaning: We have $t \in S$ so that $t(s_1 r_1' - s_1' r_1) = 0$.

We want to find $t' \in S$ so that

$$t' s_2 \Big(\underbrace{s_1'(s_2 r_1 + \cancel{s_1 r_2}) - s_1(s_2 r_1' + \cancel{s_1' r_2})}_{\parallel} \Big) = 0 \quad (\text{this is same as})$$

$$= t' s_2 \left(s_2 (s_1' r_1 - s_1 r_1') \right).$$

Take $t' = t$ and we are done.

• Multiplication is well-defined.-

To prove

$$\left.\begin{aligned} \frac{r_1}{s_1} &= \frac{r_1'}{s_1'} \\ \frac{r_2}{s_2} &= \frac{r_2'}{s_2'} \end{aligned}\right\} \Rightarrow \frac{r_1 r_2}{s_1 s_2} = \frac{r_1' r_2'}{s_1' s_2'}$$

i.e. Given $t_1, t_2 \in S$ so that

$$t_1(s_1' r_1 - s_1 r_1') = 0$$
$$t_2(s_2' r_2 - s_2 r_2') = 0$$

find $t \in S$ with $t(s_1' s_2' r_1 r_2 - s_1 s_2 r_1' r_2') = 0$

<u>Ans</u>: $t = t_1 t_2$:

$$t_1 t_2 s_1' s_2' r_1 r_2 = t_2 s_2' r_2 (t_1 s_1 r_1')$$
$$= (t_2 s_2 r_2')(t_1 s_1 r_1')$$
$$= t_1 t_2 s_1 s_2 r_1' r_2' \quad \text{as required.} \qquad \square$$

(39.5) Thus (after proving Prop. (39.4)) we obtain a new ring

Input

R: comm. ring

$S \subset R$ mult. closed

$[0 \notin S;\ 1 \in S;\ a,b \in S \Rightarrow a \cdot b \in S]$

$\rightsquigarrow$ "invert S"

Output

$$S^{-1}R = \left\{ \frac{r}{s} \;\middle|\; \begin{array}{l} r \in R, \\ s \in S \end{array} \right\} \Big/ \frac{r}{s} = 0 \Leftrightarrow \exists\, t \in S \text{ with } \boxed{t \cdot r = 0}$$

ring of fractions

e.g. (1) $R = \mathbb{Z}$ (or any integral domain

remember

R is an (integral) domain $\Leftrightarrow$ $ab = 0 \Rightarrow a = 0$ or $b = 0$

i.e. $(0) \subsetneq R$ is a prime ideal

i.e. $R \setminus \{0\} = S \subset R$ is mult. closed

$S = R \setminus \{0\}$

$\rightsquigarrow (R \setminus \{0\})^{-1} \cdot R = \mathbb{Q}$

Notation: R: integral domain

$F(R)$ "field of fractions of R"

$= (R \setminus \{0\})^{-1} R$

F for Fractions

It is a field because $x = \frac{r}{s} \in F(R) \Rightarrow$ either $r = 0$, i.e. $x = 0$

or $y = \frac{s}{r} \in F(R)$ is x^{-1}.

meaning $F(R)^{\times} = F(R) \setminus \{0\}$

(2) $R = \mathbb{Z} \supsetneq P = p\mathbb{Z}$ where $p \in \mathbb{Z}_{\geq 2}$ is prime.

[Any comm. ring $\supsetneq$ prime ideal pair will work.]

$$\mathbb{Z} \setminus p\mathbb{Z} = \left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{l} \gcd(n,p) = 1 \\ \text{or, } p \nmid n \\ \text{or, } n \notin p\mathbb{Z} \end{array} \right\} \text{ is}$$

multiplicatively closed.

$$\underset{\text{(notation)}}{\mathbb{Z}_p} := (\mathbb{Z} \setminus p\mathbb{Z})^{-1}\,\mathbb{Z} = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; \begin{array}{l} \gcd(a,b) = 1 \\ p \nmid b \end{array} \right\}$$

does not divide

In general R ← comm. ring
$\cup$
P ← prime ideal

$S = R \setminus P$ is mult closed

$(a, b \in S \Leftrightarrow a, b \notin P$
$\Leftrightarrow ab \notin P$ defn of prime ideal
$\Leftrightarrow ab \in S)$

We use the notation $R_p = (R \setminus P)^{-1} R$
"R localized at P"

Elements of R_p : $\frac{r}{s}$ where $r \in R$ and $s \notin P$

rel^n : $\frac{r}{s} = 0 \iff t \cdot r = 0$ for some $t \notin P$

Again for $x = \frac{r}{s}$ we have two cases

$r \in P$ or $r \notin P \Rightarrow \frac{s}{r} = y \in R_p$ is x^{-1}!

So $R_p^{\times} = \left\{ \frac{r}{s} \;\middle|\; \begin{array}{l} r \notin P \;\& \\ s \notin P \end{array} \right\} \Rightarrow R_p \setminus R_p^{\times} = \left\{ \frac{r}{s} \;\middle|\; \begin{array}{l} r \in P \\ s \notin P \end{array} \right\}$

$\parallel$

ideal in R_p generated by $\frac{P}{1}$

Hence $\boxed{(R_p, pR_p) \text{ is a local ring}}$