Lecture 40

(40.0) Recall we constructed rings of fractions.

$S \subset R$ mult. closed set in a comm. ring $\rightsquigarrow$ $S^{-1}R = \left\{ \frac{r}{s} \;\middle|\; \begin{matrix} r \in R \\ s \in S \end{matrix} \right\}$

$\frac{r}{s} = 0 \iff tr = 0$ for some $t \in S$.

<u>Special cases</u>

(1) R integral domain $\supset R \setminus \{0\}$ mult. closed set $\rightsquigarrow$ $F(R)$ called field of fractions of R.

(2) R : comm. ring $\supset S = R \setminus P$: mult. closed set

$R \supset P$: prime ideal

$\rightsquigarrow (R_P, P.R_P)$ a local ring, called <u>localization of R at P</u>.

(40.1) Degenerate example.

$R = \mathbb{Z}/6\mathbb{Z} \supset S = \{1, 2, 4\}$

In, $S^{-1}R$, $\frac{r}{s} = 0 \iff tr = 0$ for some $t \in S$.

So $\boxed{\frac{3}{s} = \frac{0}{s'}}$ $s, s' \in S$

$\uparrow$ means $\exists$ 6 elts in each $\sim$ in $S \times R$

$\frac{0}{s}, \frac{3}{s}$ ($s = 1, 2$ or 4)

$\Rightarrow$ There are $\frac{18}{6} = 3$ elements in $S^{-1}R$. (reps: $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}$)

(40.2) More properties of $S^{-1}R$:

1. We have a ring hom.
$$\begin{tikzcd} j: R \arrow[r] & S^{-1}R \\ r \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & \frac{r}{1} \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$
. This is clear by our construction of $S^{-1}R$.

Moreover, $\boxed{\operatorname{Ker}(j) = \{ r \mid t.r = 0 \text{ for some } t \in S \}}$

and $\boxed{j(s) = \frac{s}{1} \text{ is invertible ; i.e. in } (S^{-1}R)^{\times} \quad \forall s \in S}$

* This turns out to be "defining property" of $S^{-1}R$.

Meaning: 1^{st} iso. theorem – in a very precise sense – defined the quotient ring R/I. It says that R/I is the only ring which comes together with a $\underset{①}{\underline{\text{surjective ring}}}$ hom $\pi: R \to R/I$

and satisfies: $\left\{ \begin{array}{l} \text{every } f: R \to R' \text{ such that } \underset{②}{\underline{I \subset \operatorname{Ker}(f)}} \\ \text{factors through (uniquely!)} \end{array} \right.$
$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr] & R' \\ & R/I \arrow[u, dashed] \end{tikzcd}$$

The following result is the analogue of 1^{st} iso. theorem; for ring of fractions: $\left\{ \begin{array}{l} \text{replace ① by } j \text{ s.t } j(S) \subset (S^{-1}R)^{\times} \\ \quad \text{" ② by [illegible] } f(S) \subset (R')^{\times} \end{array} \right.$

(40.3) Prop. – Let R be a comm. ring and S a mult. closed subset. Let $f : R \longrightarrow R'$ be a ring hom such that $f(S) \subset (R')^{\times}$. Then $\exists !$ ring hom, denoted by $\tilde{f}$ below,

$$\begin{tikzcd} S^{-1}R \arrow[r, "\tilde{f}"] & R' \\ \frac{r}{s} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & f(s)^{-1} f(r) \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

such that :

Proof. – Left as an easy exercise.

(40.4) <u>Ideals in $S^{-1}R$.</u> Continue with the same assumptions on $(R, S \subset R)$; we have

$$j : R \longrightarrow S^{-1}R$$

Let $I \subset R$ be an ideal.

Lemma. (1) $\left(j(I) \right)_{S^{-1}R} = S^{-1}I \overset{(\text{defn of } S^{-1}I)}{:=} \left\{ \frac{a}{s} \;\middle|\; \begin{matrix} a \in I \\ s \in S \end{matrix} \right\}$

ideal in $S^{-1}R$ generated by $\left\{ \frac{a}{1} : a \in I \right\}$

(2) Every ideal in $S^{-1}R$ is of the form $S^{-1}I$ for some ideal $I \subset R$

(3) $S^{-1}I_1 = S^{-1}I_2$

Next class.

$\iff$ $I_1 = I_2 \pmod{\mathrm{Ker}(j)}$ ← meaning under $R \longrightarrow R/_{\mathrm{Ker}\, j}$

$I_1, I_2 \longmapsto$ same ideal

or $I_\ell \cap S \neq \phi$ $(\ell = 1, 2)$

Proof. Recall: we have the ring hom

$$\begin{array}{ccc} j: R & \longrightarrow & S^{-1}R \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ a & \longmapsto & \frac{a}{1} \end{array}$$

$$\mathrm{Ker}(j) = \left\{ a \in R \;\middle|\; \begin{array}{l} ta = 0 \\ \text{for some } t \in S \end{array} \right\} \underset{\text{ideal}}{\subsetneq} R$$

Notation: For an ideal $\tilde{I} \subset S^{-1}R$,

$$j^*\tilde{I} := \left\{ a \in R \;\middle|\; \underset{\frac{a}{1}}{\underset{\|}{j(a)}} \in \tilde{I} \right\} \subset R \text{ ideal in } R.$$

(i) $S^{-1}I \subset R$ is an ideal:

$\frac{0}{1} \in S^{-1}I$ because $0 \in I$, $1 \in S$.

$$\underset{x_1}{\underset{\|}{\boxed{\frac{a_1}{s_1}}}}, \underset{x_2}{\underset{\|}{\boxed{\frac{a_2}{s_2}}}} \in S^{-1}I \implies \begin{array}{l} s_2a_1 \pm s_1a_2 \in I \\ s_1s_2 \in S \end{array} \implies \frac{s_2a_1 \pm s_1a_2}{s_1s_2} \in S^{-1}I$$

i.e. $x_1 \pm x_2 \in S^{-1}I$.

$$\begin{array}{l} x = \frac{a}{s} \in S^{-1}I \\ y = \frac{r}{t} \in S^{-1}R \end{array} \implies yx = \frac{\overset{\nearrow\, ra \in I}{ra}}{\underset{\searrow\, st \in S}{st}} \implies yx \in S^{-1}I.$$

Hence $S^{-1}I$ is an ideal.

(ii) $\underline{\left(j(I)\right)_{S^{-1}R} = S^{-1}I}$. (given $\underline{I \subset R}$ ideal in R)

(ideal gen. by $\{ \underset{\overset{\|}{\frac{a}{1}}}{j(a)} \mid a \in I \}$)

As $\frac{a}{1} \in S^{-1}I \ \forall a \in I$, we get

$$j(I) \subset S^{-1}I$$

$$\Rightarrow \left(j(I)\right)_{S^{-1}R} \subset S^{-1}I \quad (\text{as is an } \underline{\text{ideal}})$$

Conversely $x = \frac{a}{s} \in S^{-1}I \Rightarrow s \cdot x = \frac{a}{1} \in S^{-1}I$ $\left(\begin{matrix} s \in S^{-1}R;\ x \in S^{-1}I \\ \Downarrow \\ s.x \in S^{-1}I \end{matrix} \right)$

$\Rightarrow a \in I$ and hence $S^{-1}I \subset \left(j(I)\right)_{S^{-1}R}$.

<u>Proof of (2).</u> Given: $\tilde{I} \subset S^{-1}R$ an ideal

let $I = j^{*}\tilde{I} \subset R$.

$$\begin{tikzcd} R \arrow[r, "j"] & S^{-1}R \\ \cup & \cup \\ I = j^{*}\tilde{I} \arrow[r] & \tilde{I} \end{tikzcd}$$

$$j^{*}\tilde{I} = \{ a \in R \mid \underset{\overset{\|}{\frac{a}{1}}}{j(a)} \in \tilde{I} \}$$

I is clearly an ideal.

Moreover, $S^{-1}I \underset{\text{part (1)}}{=} \left(j(I)\right)_{S^{-1}R} \subset \left(\tilde{I}\right)_{S^{-1}R} = \tilde{I}$

Conversely $\frac{r}{s} \in \tilde{I}$

$$\Rightarrow \frac{s}{1}\left(\frac{r}{s}\right) \in \tilde{I} \Rightarrow \frac{r}{1} \in \tilde{I}$$

Hence $r \in \underset{\overset{\|}{I}}{j^{*}\tilde{I}} \Rightarrow \frac{r}{s} \in S^{-1}R \Rightarrow \tilde{I} \subset S^{-1}R$.