Lecture 41

(41.0) Recall - our recent construction.

R : commutative ring

$\cup$

S : multiplicatively closed set

(i.e. $1 \in S$; $0 \notin S$; $a, b \in S \Rightarrow a \cdot b \in S$)

$\rightsquigarrow$ $S^{-1}R$: ring of fractions

&

$$j : R \longrightarrow S^{-1}R, \quad a \longmapsto \frac{a}{1}$$

a ring hom.

$[\mathrm{Ker}(j) = \{a \in R : ta = 0 \text{ for some } t \in S\}]$

Additionally, we proved :

(1) $\forall\ I \subseteq R$ ideal, the set $\left\{\frac{a}{s} : \begin{matrix} a \in I \\ s \in S\end{matrix}\right\} =: S^{-1}(I) \subset S^{-1}R$

is an ideal, and $S^{-1}(I) =$ ideal in $S^{-1}R$ generated by the subset $\{j(a) : a \in I\}$ $(= j(I))$

(2) Every ideal $\tilde{I} \subseteq S^{-1}R$ is of the form $S^{-1}(I)$. More precisely, define $j^*(\tilde{I}) \subset R$ (ideal on general grounds)

$$j^*(\tilde{I}) = \{a \in R : j(a) \in \tilde{I}\}$$

[Rk : I would rather use j^* than j^{-1} for this.]

$$\boxed{\tilde{I} = S^{-1}(j^*(\tilde{I}))}$$ $\longleftarrow$ was proved in Lecture 40.

(41.1) More properties of $\underset{\text{ideal in } R}{I} \rightsquigarrow \underset{\text{ideal in } S^{-1}R}{S^{-1}(I)}$:

$$\left.\begin{aligned} (1) \qquad S^{-1}(I_1 + I_2) &= S^{-1}(I_1) + S^{-1}(I_2) \\ S^{-1}(I_1 \cap I_2) &= S^{-1}(I_1) \cap S^{-1}(I_2) \\ S^{-1}(I_1 \cdot I_2) &= S^{-1}(I_1) \cdot S^{-1}(I_2) \end{aligned}\right\} \text{obvious from definition of } S^{-1}(I).$$

(for I_1, I_2 ideals in R)

(2) $\quad S^{-1}(I) = S^{-1}R \iff I \cap S \neq \phi$

Proof. – $(\Leftarrow)$ if $s \in I \cap S$ then $\underbrace{\frac{1}{s} \cdot s}_{= 1} \in S^{-1}(I)$

Hence $S^{-1}(I) = S^{-1}R$.

$(\Rightarrow)$ $S^{-1}(I) = S^{-1}R \Rightarrow \frac{1}{1} \in S^{-1}(I)$ i.e. $\frac{1}{1} = \frac{a}{s}$ for some $a \in I$, $s \in S$.

$\Rightarrow t(s-a) = 0$ for some $t \in S$.

i.e. $t \cdot a = s \cdot t \in S$. Also $a \in I \Rightarrow t \cdot a \in I$

$\cancel{\Rightarrow t = t \cdot \frac{1}{1} = t \cdot \frac{a}{s}}$

so $t \cdot a \in S \cap I$ hence non-empty. $\square$

(3) For any ideal $I \subset R$, we have

$$j^*(S^{-1}(I)) = \{ r \in R : tr \in I \text{ for some } t \in S \}$$

[This is almost by defn.-s: $r \in j^*(S^{-1}(I))$

$\iff (\tfrac{r}{1} =) j(r) \in S^{-1}(I)$ (defn. of $j^*(\ )$)

$\iff \dfrac{r}{1} = \dfrac{a}{s}$ for some $a \in I$, $s \in S$

$\iff s' \cdot (sr - a) = 0$ for some $a \in I$; $s, s' \in S$.

i.e. $(ss') \cdot r = s' \cdot a \in I$ $\square$]

(4)

$$\boxed{\begin{array}{ccc} \text{Prime ideals in } R & \longleftrightarrow & \text{Prime ideals in} \\ \text{not intersecting } S & & S^{-1}R \end{array}}$$

prime ideal on general grounds

$$R \supsetneq j^*(\tilde{P}) \longleftarrow\!\shortmid\ \tilde{P} \subsetneq S^{-1}R$$

prime ideal

$$P \longmapsto S^{-1}(P)$$

<u>Step 1</u>: $P \subsetneq R$ given prime ideal s.t. $P \cap S = \phi$ (to ensure $S^{-1}(P) \subsetneq S^{-1}R$) $\Rightarrow S^{-1}(P) = \left\{ \dfrac{p}{s} : \begin{array}{l} p \in P \\ s \in S \end{array} \right\}$ is a prime ideal in $S^{-1}R$.

<u>Proof</u>. Let $\dfrac{a_1}{t_1}, \dfrac{a_2}{t_2}$ be two elements of $S^{-1}R$

so that $\dfrac{a_1 a_2}{t_1 t_2} \in S^{-1}(P)$. That is, $\dfrac{a_1 a_2}{t_1 t_2} = \dfrac{p}{s}$

for some $p \in P$, $s \in S$.

$\Rightarrow \quad t(s a_1 a_2 - t_1 t_2 p) = 0$ for some $t \in S$.

i.e. $\quad t s\, a_1 a_2 = (t t_1 t_2) p \in P$ as $p \in P$.

Since P is prime & $ts \in S$, $P \cap S = \phi$

$\Rightarrow a_1 \cdot a_2 \in P \Rightarrow a_1$ or $a_2 \in P$

Hence either $\frac{a_1}{s_1} \in S^{-1}(P)$ or $\frac{a_2}{s_2} \in S^{-1}(P)$, proving that $S^{-1}(P)$ is a prime ideal. □

<u>Step 2.</u> – $\begin{matrix} P \subsetneq R \text{ prime} \\ P \cap S = \phi \end{matrix} \Rightarrow j^*(S^{-1}(P)) = P$

Pf. use (3) from previous page: $r \in j^*(S^{-1}(P)) \Longleftrightarrow$ $t \cdot r \in P$ for some $t \in S$. Thus $t \notin P$ and $t \cdot r \in P$

$\Rightarrow$ (as P is prime) $r \in P$. [In fact: $\forall\, t \in S$:

$t \cdot r \in P \Longleftrightarrow r \in P$
because $t \notin P$.] □

Finally $S^{-1}(j^*(\tilde{P})) = \tilde{P}$ because it is true for any ideal $\tilde{I}$ (see page 1.)

Hence
$$\begin{matrix} P & \longmapsto & S^{-1}(P) \\ j^*\tilde{P} & \longleftarrow & \tilde{P} \end{matrix}$$
are inverse to each other. □

(41.2) $\left\{ \text{Same set up:} \quad \begin{array}{l} R \text{ is a comm. ring.} \\ S \subset R \text{ a mult. closed set} \end{array} \right\}$.

$$\begin{array}{cccc} j: & R & \longrightarrow & S^{-1}R \\ & \cup & & \cup \ \text{ideal} \\ & j^*\tilde{I} & \dashrightarrow & \tilde{I} \\ & \| & & \end{array} \qquad\qquad \begin{array}{cccc} j: & R & \longrightarrow & S^{-1}R \\ & \cup & & \cup \ \text{ideal} \\ & I & \rightsquigarrow & S^{-1}(I) \\ & \text{ideal} & & \end{array}$$

$\{r \in R \mid j(r) \in \tilde{I}\}$ ideal in R

Item (3) on page 3: $\quad j^*(S^{-1}(I)) = \{r \in R \mid tr \in I \text{ for some } t \in S\}$

$$\begin{array}{c} \cup \\ I \quad (\text{take } t = 1 \in S) \end{array}$$

Example of $\quad I \subset R \quad$ s.t. $\underline{j^*(S^{-1}(I)) \neq I \ \& \ I \cap S = \phi}$

(note - see page (1) - $\quad S^{-1}(I) = S^{-1}(j^*(S^{-1}(I)))$

so we will have two distinct ideals

$I \ \& \ j^*(S^{-1}(I))$ (in R)

<u>not intersecting S</u>, ~~so~~ which generate

the same ideal in $S^{-1}R$, via j.)

Of course, by (4) on page 3, we should not take I = any prime ideal in our example.

Take $\quad R = K[x,y]$; $\quad S = R \setminus (x)$; $\quad I = (x \cdot y)$

K a field

$S = \{f(x,y) \mid f \text{ is } \underline{\text{not}} \text{ divisible by } x\}$

$$S^{-1}R = \left\{ \frac{f(x,y)}{g(x,y)} : g \text{ is not div. by } x \right\}$$

As $y \in S$; $y \cdot x \in I \Rightarrow x \in j^*(S^{-1}(I))$

but $x \notin I$.

<u>Ex.</u> $j^*(S^{-1}(I)) = (x) \quad \left(\supsetneq (x \cdot y) = I \right)$.

(41.3) An application : <u>Nil radical of R</u>. (R: comm. ring).

Recall that $a \in R$ is called <u>nilpotent</u> if $\exists\ n \geqslant 0$ s.t. $a^n = 0$.

$$\mathcal{N} := \{ a \in R : a \text{ is nilpotent} \}$$

Remark: as we do not allow 0 to be in our mult. closed set, <u>for a given $a \in R$, the set $\{1, a, a^2, a^3, \dots\}$ is mult. closed $\Leftrightarrow a \notin \mathcal{N}$.</u>

<u>Prop.</u> (1) $\mathcal{N} \subsetneq R$ is an ideal.

(2) $\mathcal{N} \subset P$ for any prime ideal $P \subsetneq R$.

(3)
$$\bigcap_{\substack{P \subsetneq R \text{ is} \\ \text{a prime ideal}}} P = \mathcal{N}$$

(called <u>nil radical</u> of R).

Proof (1): was a homework problem.

(2): $a \in \mathcal{N} \Rightarrow a^n = 0 \in P$ for $P \subsetneq R$ a prime ideal.

P: Prime $\Rightarrow a \in P$.

(3): If $a \notin \mathcal{N}$, then $S = \{1, a, a^2, \dots\} \subset R$ is a mult. closed set. Consider the ring hom.

$$j : R \longrightarrow S^{-1}R$$

Take any prime ideal $\tilde{P} \subsetneq S^{-1}R$ (exists - because maximal ideals exist; and maximal $\Rightarrow$ prime.)

and set $P = j^*(\tilde{P}) \subsetneq R$ prime ideal not intersecting $S = \{1, a, a^2, \dots\}$. Hence $a \notin P$.

We have thus shown: $a \notin \mathcal{N} \Rightarrow \exists$ prime ideal $P \subsetneq R$ so that $a \notin P$.

Hence $\displaystyle\bigcap_{\substack{P:\text{ prime ideal}\\ \text{in } R}} P = \mathcal{N}$ as required. $\square$