Lecture 42

(42.0) Recall : basic operations with rings and their ideals

Input	Output	Ideals
R_1, R_2 : two rings (comm.)	$R_1 \times R_2$ (direct product) & ring hom-s. $R_1 \longleftarrow R_1 \times R_2 \longrightarrow R_2$	Ideals of $R_1 \times R_2$ $= \left\{ I_1 \times I_2 \;\middle|\; \begin{array}{l} I_1 \subset R_1 \\ I_2 \subset R_2 \end{array} \right\}$ ideals
R : (comm) ring	(& ring hom-s) $R[x] \longleftrightarrow R$ or $R[[x]] \longleftrightarrow R$ (constants)	• When $R = K$ is a field, we know every ideal is principal & $K[[x]]$ is local w/ max'l ideal (x).
R comm ring $\supsetneq I$ ideal (proper)	R/I : quotient ring $\pi : R \longrightarrow R/I$ ring hom	Ideals of R/I $= \left\{ \underset{\pi(K)}{\overset{\Vert}{K/I}} : \begin{array}{l} K \subseteq R \text{ ideal} \\ \text{s.t. } I \subseteq K \end{array} \right\}$
R : comm. ring; $S \subset R$: mult. closed set (i.e. $0 \notin S$; $1 \in S$; $a, b \in S \Rightarrow a \cdot b \in S$)	$S^{-1}R$: ring of fractions; $j : R \longrightarrow S^{-1}R$: ring hom	$\underset{\text{ideal}}{I \subseteq R} \rightsquigarrow S^{-1}I \subseteq S^{-1}R$, $S^{-1}I = \left\{ \frac{a}{s} : a \in I, s \in S \right\}$; $j^*(\tilde{I}) \subset R$ ideal $\leftsquigarrow \tilde{I} \subseteq S^{-1}R$ ideal; $j^*(\tilde{I}) = \{ a \in R \mid j(a) \in \tilde{I} \}$

(42.1) Let R be a commutative ring. Define

$$\mathcal{J} := \bigcap_{\substack{M \subsetneq R \\ \text{maximal ideal}}} M \qquad (\text{Jacobson radical of } R)$$

• Recall - analogy - $\mathcal{N} = \bigcap_{\substack{P \subsetneq R \\ \text{prime ideal}}} P$ = ideal of nilpotent elements i.e. <u>nilradical</u> of R

Since $M \subsetneq R$ maximal ideal $\Rightarrow$ M is a prime ideal

Hence, $\boxed{\mathcal{N} \subset \mathcal{J}}$.

Prop. $\mathcal{J} = \{a \in R : 1 - xa \in R^\times \ \forall x \in R\}$

Proof. - Let us denote the set $\{a \in R : 1 - xa \in R^\times \ \forall x \in R\}$ by $K \overset{\text{defn.}}{:=}$

We want to prove that $\mathcal{J} = K$.

<u>$\mathcal{J} \subset K$</u> : ~~To prove $a \in M$, $\forall\, M \subsetneq R$ maximal ideal $\Rightarrow 1 - xa \in R^\times$ $\forall x \in R$~~

~~Let us assume $1 - ya \notin R^\times$ for some $y \in R$.~~

~~But that means $(1-ya) \subset R$ is a proper ideal. Since~~

Proof of Prop:

We want to prove, for any given element $a \in R$:

$$1 - ax \in R^\times \;\; \forall x \in R \iff a \in M, \; \forall \text{ maximal ideal } M \subsetneq R.$$

Equivalently,

There is a maximal ideal $M \subsetneq R$ s.t. $a \notin M$ $\iff$ There is some element $y \in R$ s.t. $1-ay$ is <u>not</u> a unit.

($\Rightarrow$) Since $a \notin M$ & M is maximal, we get $\boxed{(M, a)} = R$

ideal gen. by M and a

i.e. $1 = m + ra$ for some $m \in M$ and $r \in R$

i.e. $1 + (-r)a = m \in M$ hence cannot be a unit.

($\Leftarrow$) $1 - ay \notin R^\times \Rightarrow (1-ay) \subsetneq R$

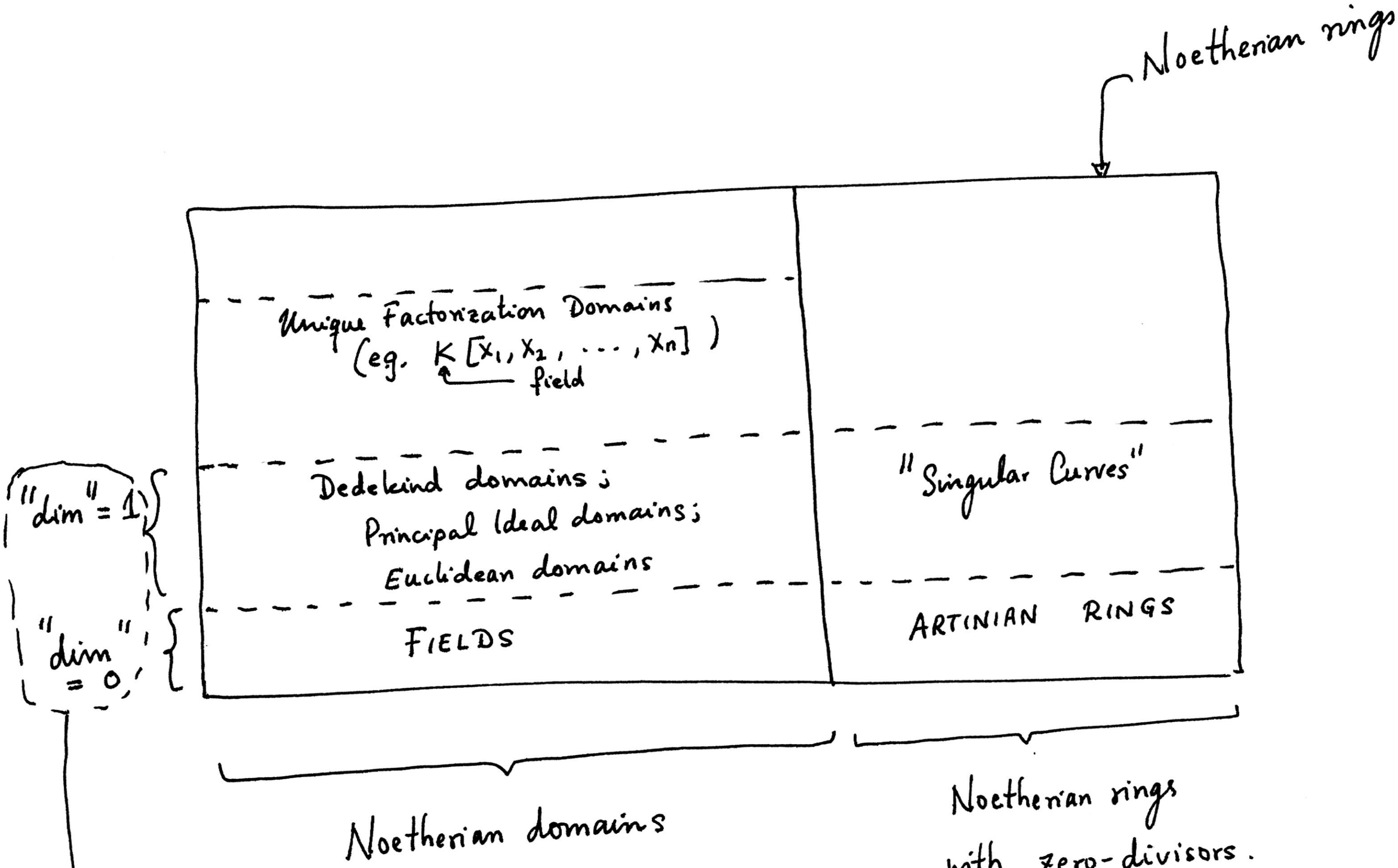
$\Rightarrow$ $\exists$ a maximal ideal $M \subsetneq R$ s.t. $(1-ay) \subset M$.

(see Prop 37.3 on page 3 of Lecture 37)

But then $a \notin M$, as both $1-ay$, $a \in M$ $\Rightarrow 1 \in M$ which is not true.

$\square$

(42.2) Landscape of rings to be studied



"dim" $(R) = 0$ means every prime ideal is maximal.

"dim" $(R) = 1$ & R is an integral domain } means every non-zero prime ideal is maximal.