Lecture 43

(43.0) Let R be a commutative ring. We say that R is <u>Noetherian</u> (after Emmy Noether) if the following holds:

Ascending Chain Condition

> Given any ascending chain of ideals in R
> $$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$
> there exists $n \geq 1$ such that $I_n = I_{n+1} = \cdots$

<u>Non-example</u>: Let R = ring of continuous functions (real-valued) of one (real) variable x

$$= \{ f : \mathbb{R} \to \mathbb{R} \text{ continuous} \}$$

I_n = $f \in R$ such that $f(x) = 0 \ \forall x \in \left[\frac{-1}{n}, \frac{1}{n}\right]$

$(n = 1, 2, 3, \ldots)$

As $[-1, 1] \supset \left[\frac{-1}{2}, \frac{1}{2}\right] \supset \left[\frac{-1}{3}, \frac{1}{3}\right] \supset \cdots$

we get $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$

[Ex.] This chain never stops (i.e. "stabilizes" as in the condition defining Noetherian rings).

(43.1) Before giving examples of Noetherian rings, we need the following 2 equivalent ways of proving if

R is Noetherian.

Prop. Again, R is a commutative ring. Then the following are equivalent:

(1) <u>R is Noetherian.</u>

(2) Given an ideal $I \subseteq R$, there exist finitely many $a_1, a_2, \dots, a_N \in I$ s.t. $I = (a_1, \dots, a_k)$.

[Read: <u>every ideal in R is finitely generated.</u>]

(3) Let X be a non-empty set of ideals of R. Then

$\exists\, J \in X$ such that $\boxed{\begin{array}{l} J \subseteq I \\ \& \ I \in X \end{array} \Rightarrow J = I}$

[Read: <u>every set of ideals in R has a maximal</u>, with respect to inclusion, <u>ideal</u>.]

Proof. We will check that $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$.

<u>$(1) \Rightarrow (3)$</u>: We assume that R is Noetherian, i.e. ascending chain condition holds for R.

Let $\cancel{I \subseteq}$ X be a non-empty set of Ideals in R.

We can start forming a chain (ascending) of ideals from X as follows:

Choose $I_1 \in X$. If I_1 is maximal among all the ideals from X, then we are done. Otherwise, pick $I_2 \in X$ s.t. $I_1 \subsetneq I_2$. Repeat the same argument with I_2, and continue:

we get: $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ each $I_j \in X$.

By definition, this process must stop after some finite number of iteration. Meaning, for some $N \geq 1$, $I_N \in X$ is maximal among all the ideals from X. This is exactly what we wanted to prove.

<u>(3) ⇒ (2)</u> :- Let $I \subseteq R$ be an ideal. Consider the following set of ideals in R:

$$X = \left\{ I' \subset I \ : \ \begin{array}{l} I' \text{ is a finitely generated} \\ \text{ideal of } R \end{array} \right\}$$

$X \neq \phi$, say for instance $(0) \in X$.

exists, because of (3)

Let $I_1 \in X$ be maximal among all ideals from X. Then

- $I_1 \subset I$
- I_1 is finitely generated

- $I_1 \subset I_2 \subset I$
  I_2 also finitely generated
  $\Rightarrow I_1 = I_2$.

<u>We claim that $I_1 = I$</u>, and hence I is finitely generated.

↳ if not, $\exists\, a \in I \setminus I_1$. Take $I_2 =$ ideal gen. by I_1 & a.

Then $I_2 \supsetneq I_1$; I_2 is also finitely generated. This contradicts
& $I_2 \subseteq I$

maximality of I_1 and hence $I_1 = I$ as claimed.

(2) $\Rightarrow$ (1) : Assume we are given an ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

Take $I = \bigcup_{j \geq 1} I_j \subset R$. Then I is an ideal.
(see page 5 of Lecture 37.)

By our hypothesis (statement (2) of the proposition), $I = (a_1, \ldots, a_N)$
for some finite number of elements $a_1, \ldots, a_N \in I$.

$\Rightarrow \exists\ k_1, k_2, \ldots, k_N$ s.t.

$$a_1 \in I_{k_1}$$
$$a_2 \in I_{k_2}$$
$$\vdots$$
$$a_N \in I_{k_N}$$

$\Rightarrow$ (take $M = \max\{k_1, \ldots, k_N\}$)

$$a_1, a_2, \ldots, a_N \in I_M \subseteq I_{M+1} \subseteq \cdots$$

$\Rightarrow I_M = I_{M+1} = \cdots = I$, as we wanted to prove.

$\square$

(43.2) More examples:

(1) Every principal ideal ring is Noetherian.

(recall: R is principal ideal ring if every ideal I is of the form $I = (a)$ (principal))

e.g. $R = K$: any field ; $\mathbb{Z}$; $K[x]$; $K[[x]]$
$\uparrow$ any field

$\mathbb{Z}/n\mathbb{Z}$ are all Noetherian.
$(n \geq 2)$

(2) If $R = K[x_1, x_2, x_3, \dots]$ polynomial ring in infinitely many variables, then R is <u>not</u> Noetherian, since $I = (x_1, x_2, x_3, \dots)$ can not be generated by finitely many elements.

(43.3) <u>Being Noetherian</u> vs. <u>our basic operations with rings.</u>

Prop. R : a commutative ring. Assume R is Noetherian.

(1) For any proper ideal $I \subsetneq R$, the quotient ring R/I is Noetherian.

(2) For any mult. closed set $S \subset R$, the ring of fractions $S^{-1}R$ is Noetherian

(3) [easiest*] R_1, R_2 Noetherian $\Rightarrow$ $R_1 \times R_2$ is Noetherian

Proof. (1) (see page 6 of Lecture 35).

We want to prove that every ideal of R/I is finitely generated. Let $\tilde{I} \subset R/I$ be an ideal. We know that

$\tilde{I} = \pi(J)$ for some ideal $J \subset R$; here $\pi : R \to R/I$ is the usual projection. Since R is Noetherian, J is finitely generated, say $J = (a_1, a_2, \ldots, a_N)$. Then $\tilde{I} = (\pi(a_1), \ldots, \pi(a_N))$ is also finitely generated, as we wanted to prove.

(2) (same logic – see page 3 - Lemma 40.4 Lecture 40).

Again, we will prove that every ideal in $S^{-1}R$ is finitely generated. Since ideals in $S^{-1}R$ are of the form

$$S^{-1}I = \left\{ \frac{a}{s} : a \in I, s \in S \right\}$$

where $I \subseteq R$ is an ideal, we know I is finitely generated, say $I = (b_1, \ldots, b_\ell)$ (in R). Then $S^{-1}I = \left(\frac{b_1}{1}, \frac{b_2}{1}, \ldots, \frac{b_\ell}{1} \right)$ in $S^{-1}R$ is also finitely generated.

(3) Left as an exercise. $\square$

(43.4) <u>Subring of a Noetherian ring need not be Noetherian</u>

Example: $R = K[x_1, x_2, x_3, \ldots]$ is <u>not</u> Noetherian, (K: field) but still, it is an integral domain. So we can form its field of fractions : see Lecture 40 page 1.

$$j : R \longrightarrow F(R) \quad \text{injective.}$$

$F(R)$: a field hence <u>Noetherian</u>.

R: subring of $F(R)$ but <u>not</u> Noetherian.