Lecture 44

(44.0) Recall: we defined Noetherian rings as commutative rings satisfying either one of the following 3 properties (we proved that they are equivalent).

(1) Ascending Chain Condition: Given any ascending chain of ideals
$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$
we can find $n \geq 1$ so that $\underbrace{I_n = I_{n+1} = I_{n+2} = \cdots}$
(i.e., it stabilizes)

(2) Every non-empty set of ideals has at least one maximal element

(3) Every ideal is finitely generated.

We also showed that

$$R : \text{Noetherian} \Longrightarrow R/I \text{ is Noetherian}$$
($\forall$ proper ideal $I \subsetneq R$)

$$\Downarrow$$

$S^{-1}R$ is Noetherian
$\forall$ mult. closed set $S \subset R$.

(44.1) Hilbert Basis Theorem.

$$R : \text{Noetherian} \Longrightarrow R[x] \text{ is Noetherian.}$$

Some notations and terms for polynomials:

$$f(x) = a_0 + a_1 x + \dots + a_{n-1} x^{n-1} + a_n x^n \; ; \quad \text{assume } n \text{ is largest so that } a_n \neq 0$$

- $n = \underline{\text{degree}(f)}$ $\underline{\text{degree of } f(x)}$
- $a_0 = f(0)$ $\underline{\text{constant term of } f(x)}$
- $a_n = \underline{\text{Leading Coefficient of } f(x)}$, let us denote it by $\underline{L(f)}$

[Convention: $L(\text{zero poly.}) = 0$].

(44.2) <u>Ideal of Leading Coefficients</u>: Let R be a comm. ring.

Given $\tilde{I} \subseteq R[x]$ an ideal; consider the set

$$L(\tilde{I}) \subset R$$
$$\| $$
$$\{a \in R \text{ such that } a = L(f) \text{ for some } f(x) \in \tilde{I}\}$$

Lemma. $L(\tilde{I}) \subset R$ is an ideal.

Proof. $0 = L(\text{Zero poly.}) \Rightarrow 0 \in L(\tilde{I})$.

Let $a \in L(\tilde{I})$, $r \in R$. Then, assuming $ra \neq 0$, $r \cdot a = L(r \cdot f)$ if $a = L(f)$. As $f(x) \in \tilde{I}$; $r \cdot f(x) \in \tilde{I}$, hence $r \cdot a \in L(\tilde{I})$. (if $r \cdot a = 0$ we have nothing to prove).

Thus $R \cdot L(\tilde{I}) \subset L(\tilde{I})$. <u>It remains to show that</u>

$a, b \in L(\tilde{I}) \implies a+b \in L(\tilde{I})$ [for $a-c$, take $b=-c$ as $c \in L(\tilde{I}) \Rightarrow (-1)\cdot c \in L(\tilde{I})$ is already proven.]

We may assume $a+b \neq 0$.

We know $a = L(f)$, $b = L(g)$ for some $f, g \in \tilde{I}$.

Let $n = \text{degree}(f)$, $m = \text{degree}(g)$ and without sacrificing any generality, assume $n \leq m$.

Then $a+b = L(\underbrace{x^{m-n} \cdot f + g})$ $\in \cancel{L(\tilde{I})}\ \tilde{I}$ because $f(x), g(x) \in \tilde{I}$.

$\implies a+b \in L(\tilde{I})$.

$\square$

(44.3) Baby steps – towards a proof of Hilbert Basis Theorem

Prop. If R is a Noetherian ring and $D \in \mathbb{Z}_{\geq 1}$, then

$$R[X]/(X^D)$$

is also Noetherian.

Remark. – This statement in fact would follow from Hilbert Basis Theorem, but <u>we are going to use it in our proof</u>.

We will in fact prove a stronger result. Namely.

Lemma: Given an abelian subgroup $J \subset R[x]/(x^D)$ such that $R.J \subset J$; we can find finitely many $f_1(x), \dots, f_p(x) \in J$ such that $J = R.f_1(x) + R.f_2(x) + \dots + R f_p(x)$.

Proof. For each $k \in \{0, \dots, D-1\}$ define

$$C_k(J) = \{0\} \cup \left\{ a \in R \text{ such that there is } f(x) \in J \text{ of the form } f(x) = a.x^k + \boxed{\text{terms involving } x^{k+1} \dots x^{D-1}} \right\}$$

[Ex.] $C_k(J) \subset R$ is an ideal by our assumptions on J (i.e. $J \subset R[x]/(x^D)$ is a subgp. and $R.J \subset J$)

As R is Noetherian, $C_k(J) = (\alpha_1^{(k)}, \dots, \alpha_{m_k}^{(k)})$ for some finite number of elements $\alpha_1^{(k)}, \dots, \alpha_{m_k}^{(k)} \in R$.

These elements come from $f_1^{(k)}(x), \dots, f_{m_k}^{(k)}(x) \in J$.

That is,
$$f_1^{(k)}(x) = \alpha_1^{(k)} . x^k + \boxed{\text{terms involving } x^{k+1}, \dots, x^{D-1}} \in J$$
$$\vdots$$
$$f_{m_k}^{(k)}(x) = \alpha_{m_k}^{(k)} . x^k + \boxed{\text{terms involving } x^{k+1}, \dots, x^{D-1}} \in J$$

Claim: $J = \left(R\cdot f_1^{(0)}(x) + \dots + R f_{m_0}^{(0)}(x)\right)$
$+ \left(R f_1^{(1)}(x) + \dots + R f_{m_1}^{(1)}(x)\right)$
$+ \dots + \left(R f_1^{(D-1)}(x) + \dots + R f_{m_{D-1}}^{(D-1)}(x)\right)$

Proof. If $g(x) = \gamma \cdot x^{\ell} + \boxed{\text{terms involving } x^{\ell+1}, \dots, x^{D-1}} \in J$

then $\gamma \in C_\ell(J) = (\alpha_1^{(\ell)}, \dots, \alpha_{m_\ell}^{(\ell)}) \subset R$. Meaning

$\gamma = r_1 \alpha_1^{(\ell)} + \dots + r_{m_\ell} \alpha_{m_\ell}^{(\ell)}$ for some $r_1, \dots, r_{m_\ell} \in R$.

$$\Rightarrow \bar{g}(x) = g(x) - \sum_{j=1}^{m_\ell} r_j \cancel{\alpha_j^{(\ell)}} f_j^{(\ell)} \in J \text{ ~~than~~}$$

and $\bar{g}(x) = \bar{\gamma}\, x^{\ell+1} + \boxed{\text{terms involving } x^{\ell+2}, \dots, x^{D-1}}$

Repeat the same argument w/ $\begin{cases} \bar{\gamma} \text{ replaces } \gamma \\ \ell+1 \text{ replaces } \ell \end{cases}$.

After $D-\ell$ iterations of this argument – we get the claim. $\square$

(44.4) Proof of Hilbert Basis Theorem. Now let R be a Noetherian ring and $\tilde{I} \subset R[x]$ be an ideal. We want to prove that $\tilde{I}$ is finitely generated.

~~finitely generated~~.

Step 1. Take $I = L(\tilde{I}) \subset R$ ideal by Lemma 44.2.

Hence finitely generated, since R is Noetherian.

So, $I = (a_1, \dots, a_N)$ and therefore we get

$$g_1(x) = a_1 x^{d_1} + \underbrace{\boxed{}}_{\text{terms involving } x^{d_1-1}, \dots, x^0} \in \tilde{I}$$

$$\vdots$$

$$g_N(x) = a_N x^{d_N} + \underbrace{\boxed{}}_{\text{terms involving } x^{d_N-1}, \dots, x^0} \in \tilde{I}$$

Step 2 (Division algorithm) degrees of $g_1, \dots, g_N$ respectively.

If $D = \max\{d_1, \dots, d_N\}$, then $\forall\, g(x) \in \tilde{I}$,

$\exists\ \bar{g}(x) \in \tilde{I}$ s.t. degree $(\bar{g}) < D$

$g = \bar{g} \mod (g_1, \dots, g_N)$

Proof. If degree$(g) < D$ we have nothing to prove.

If not, then $g(x) = \gamma \cdot x^M + \boxed{\text{degree} < M}$ & $M \geq d_j\ (\forall j)$

As $\gamma \in I = (a_1, \dots, a_N)$ we have $r_1, r_2, \dots, r_N \in R$

s.t. $\gamma = r_1 a_1 + \dots + r_N a_N$

$$\Rightarrow \quad g(x) - \sum_{j=1}^{N} r_j \, g_j(x) \, x^{M-d_j} \in \tilde{I}$$

has degree $< M$. Repeat until degree$(\bar{g}) < D$.

Final Step: (Take care of polynomials in $\tilde{I}$ of degree $< D$ using Lemma 44.3 on page 4 above.)

Let us denote by $\tilde{I}_{<D} = \{ f(x) \in \tilde{I} \mid \text{degree}(f) < D \}$.

Then $\tilde{I}_{<D} \subset R[x]$ is an abelian subgroup and

$$R \cdot \tilde{I}_{<D} \subset \tilde{I}_{<D}.$$

By the argument from the proof of Lemma 44.3 (page 4):

$\exists \; f_1(x), \ldots, f_p(x) \in \tilde{I}_{<D}$ s.t.

$$\tilde{I}_{<D} = R \cdot f_1(x) + R f_2(x) + \cdots + R f_p(x).$$

Hence (combining w/ the conclusion of Step 2)

$$\tilde{I} = (g_1, \ldots, g_N ; f_1, \ldots, f_p)$$ is finitely

generated. $\square$