Lecture 45

Primary decomposition of ideals in a Noetherian (commutative) ring R.

(45.0) Some definitions

1. An ideal $Q \subsetneq R$ is called <u>primary</u> if it is <u>proper</u> and

$$\begin{matrix} a \cdot b \in Q \\ \& \ a \notin Q \end{matrix} \Rightarrow b^n \in Q \text{ for some } n \geq 1.$$

[In particular, every prime ideal is primary.]

2. An ideal $I \subset R$ is said to be <u>an irreducible ideal</u> if

$$\begin{matrix} I = I_1 \cap I_2 \\ \text{for some ideals} \\ I_1, I_2 \subset R \end{matrix} \Rightarrow \begin{matrix} I = I_1 \text{ or} \\ I = I_2 \end{matrix}$$

[Every prime ideal is irreducible. <u>Proof</u>: Let $P \subsetneq R$ be a prime ideal and assume $I_1, I_2 \subset R$ are ideals so that

$P = I_1 \cap I_2$. (Therefore, $P \subset I_1$ & $P \subset I_2$.)

Then $I_1 \cdot I_2 \subset I_1 \cap I_2 = P \Rightarrow I_1 \subset P$ or $I_2 \subset P$ [from a homework problem.]

. Hence either $I_1 = P$ or $I_2 = P$. $\square$].

3. Let $I \subset R$ be an ideal. The radical of I, denoted by $\mathrm{Rad}(I)$, is defined as

$$\mathrm{Rad}(I) = \{ a \in R \mid a^n \in I \text{ for some } n \geq 1 \} \subset R.$$

[Consider the ring hom. $\pi : R \longrightarrow R/I$.
Let $\overline{N} \subset R/I$ be the nilradical (i.e. the set of nilpotent elements of R/I - which, we proved that, was an ideal.).

Then $\mathrm{Rad}(I) = \pi^*(\overline{N}) = \{ a \in R \mid \pi(a) \in \overline{N} \}$.
This way, we see directly why $\mathrm{Rad}(I) \subset R$ is an ideal.]

(45.1) Some easy results :

(1) $Q \subsetneq R$ primary ideal $\Rightarrow$ $P = \mathrm{Rad}(Q) \subsetneq R$ is a prime ideal.

Pf : To show : given $a, b \in R$ s.t. $a \cdot b \in P$, $a \notin P$, we have $b \in P$.
(by defn. of a prime ideal)

$ab \in P \Rightarrow a^n b^n \in Q$ for some $n \geq 1$

$a \notin P \Rightarrow a^m \notin Q$ for every $m = 0, 1, 2, \ldots$

Thus $a^n b^n \in Q$ & $a^n \notin Q$ $\Rightarrow (b^n)^\ell \in Q$ for some $\ell \geq 1 \Rightarrow b \in \underset{\overset{\parallel}{P}}{\mathrm{Rad}(Q)}$

$\square$

(2) $Q \subsetneq R$ is a primary ideal

$$\iff \text{Zero-divisors of } R/Q = \text{Nilpotent elements of } R/Q.$$

(Proof is left as an easy exercise.)

(3) $P \subsetneq R$ prime ideal $\Rightarrow$ $\mathrm{Rad}(P^n) = P$, for each $n \geq 1$.

$\boxed{I \subset P \Rightarrow \mathrm{Rad}(I) \subset P}$ for any ideal $I \subset R$.

Proof: $a \in \mathrm{Rad}(I) \Rightarrow a^n \in I$ for some $n \geq 1$

$\Rightarrow a^n \in P$ (as $I \subset P$) $\Rightarrow a \in P$ (as P is a prime ideal). $\square$

Now $P^n \subset P$ for every $n \geq 1$ $\Rightarrow$ $\mathrm{Rad}(P^n) \subset P$.
(P: prime ideal)

However, $\forall x \in P$, $x^n \in P^n \Rightarrow x \in \mathrm{Rad}(P^n)$

Thus $\mathrm{Rad}(P^n) = P$. $\square$

(4) If $Q_1, \ldots, Q_\ell \subsetneq R$ are primary ideals so that

$$\mathrm{Rad}(Q_1) = \cdots = \mathrm{Rad}(Q_\ell) = P$$

Then $Q = Q_1 \cap \cdots \cap Q_\ell$ is also primary and $\mathrm{Rad}(Q) = P$.

Proof Let us prove first that Q is primary. Assume we have

$a, b \in R$ s.t. $a \cdot b \in Q$, $a \notin Q$; i.e. $ab \in Q_j$ for <u>every</u> $1 \leq j \leq \ell$, $a \notin Q_k$ for <u>some</u> $k \in \{1, \ldots, \ell\}$

In particular, $\begin{matrix} ab \in Q_k \\ a \notin Q_k \end{matrix}$ & Q_k is a primary ideal, hence,

$a^n \in Q_k$ for some $n \geq 1$ $\Rightarrow$ $a \in Rad(Q_k) = P$.

As $P = Rad(Q_j)$ for every $j \in \{1, \dots, \ell\}$, we get that $\exists\, N_j \geq 1$ s.t. $a^{N_j} \in Q_j$. Thus $N = \max\{N_1, \dots, N_\ell\}$ gives $a^N \in Q_j$ for every $j \in \{1, \dots, \ell\}$. $\Rightarrow a^N \in Q$.

Hence, Q is a primary ideal. $Rad(Q) = P$ because:

(5): $\boxed{Rad(I_1 \cap I_2) = Rad(I_1) \cap Rad(I_2)}$

Pf: $a \in Rad(I_1) \cap Rad(I_2)$ $\Leftrightarrow$ $\exists\, n_1 \geq 1$ and $n_2 \geq 1$ such that $a^{n_1} \in I_1$ and $a^{n_2} \in I_2$

$\Rightarrow a^{\max(n_1, n_2)} \in I_1 \cap I_2$

$\Rightarrow a \in Rad(I_1 \cap I_2)$

$a \in Rad(I_1 \cap I_2)$ $\Leftrightarrow$ $\exists\, n \geq 1$ such that $a^n \in I_1 \cap I_2$

$\Leftrightarrow a^n \in I_1$ & $a^n \in I_2$

$\Rightarrow a \in Rad(I_1) \cap Rad(I_2)$. $\square$

(45.2) One difficult property.

Prop: R: Noetherian $\Rightarrow$ every (proper) irreducible ideal is primary.

Proof: Let $I \subsetneq R$ be a proper ideal. Let us assume that I is irreducible. We want to prove:

$$\left.\begin{array}{l} ab \in I \\ a \notin I \end{array}\right\} \Rightarrow b^n \in I \text{ for some } n \geq 1.$$

Introduce: $\boxed{(I : b^k) \underset{\text{defn.}}{=} \{x \in R \mid x \cdot b^k \in I\}.}$

As I is an ideal; $x b^k \in I \Rightarrow x b^{k+1} \in I$; resulting in an ascending chain:

$$\underset{\substack{\| \\ I \not\ni a}}{(I : b^0 = 1)} \subsetneq \underset{\substack{\cup \\ a}}{(I : b)} \subset (I : b^2) \subset \cdots$$

As R is Noetherian, $\exists\, n \geq 1$ s.t. $(I : b^n) = (I : b^{n+1})$.

<u>Claim.</u> $I = \underbrace{(I + R.a)}_{\text{ideal generated by } I \text{ and } a} \cap \overbrace{(I + Rb^n)}^{\text{ideal gen by } I \,\&\, b^n}$

As $\begin{array}{l} I \subset (I + R.a) \\ I \subset (I + Rb^k) \\ \quad \forall\, k \geq 0 \end{array}$ we only need to prove that ——— next page:

$\cancel{\xi \in (I + R.a) \cap (I + Rb^n) \Rightarrow \xi \in I}$.

$\cancel{\xi \in (I + Ra) \Rightarrow b\xi \in I}$

$$\xi \in (I + Ra) \cap (I + R \cdot b^n) \Rightarrow \xi \in I.$$

Now $\xi \in I + Ra$ & $a \cdot b \in I \Rightarrow b \cdot \xi \in I$.

$\xi \in I + R \cdot b^n \Rightarrow \xi = \xi_0 + c \cdot b^n$ for some $\xi_0 \in I$, $c \in R$.

Combining the two, we get $b \cdot (\xi_0 + c b^n) \in I$

$\Rightarrow c \cdot b^{n+1} \in I \Rightarrow c \in (I : b^{n+1}) = (I : b^n)$ (see last page)

Hence $c \cdot b^n \in I$

$\Rightarrow \xi = \xi_0 + c \cdot b^n \in I$ as we wanted.

Since $a \notin I$, we get $I \neq I + R \cdot a$. Being irreducible, the only option for I is $I = I + R \cdot b^n$; i.e. $b^n \in I$ as we wanted to prove. $\square$

[Lasker – Noether for $K[X_1, \ldots, X_n]$ (K field) 1905, in general – as below 1921]

(45.3) **Theorem 1.** Let R be a commutative Noetherian ring. Let $I \subsetneq R$ be a proper ideal. Then there exist <u>primary ideals</u> $Q_1, \ldots, Q_\ell \subsetneq R$ s.t.

(1) $I = Q_1 \cap \cdots \cap Q_\ell$

(2) $P_1 = \mathrm{Rad}(Q_1), \ldots, P_\ell = \mathrm{Rad}(Q_\ell)$ are all distinct

(3) $I = Q_1 \cap \dots \cap Q_\ell$ has no irrelevant terms.

That is, for every $i \in \{1, \dots, \ell\}$,

$$Q_i \not\supset \bigcap_{\substack{j \in \{1,\dots,\ell\} \\ j \neq i}} Q_j$$

[Meaning - we cannot just omit Q_i from $I = Q_1 \cap \dots \cap Q_\ell$.]

Proof We begin by showing that every proper ideal in a Noetherian ring can be written as an intersection of finitely many irreducible ideals.

For the sake of contradiction, assume that the following set is non empty: $\sum = \left\{ I \underset{\text{ideal}}{\subsetneq} R \;\middle|\; \begin{array}{l} I \text{ cannot be written as} \\ \text{any finite intersection} \\ \text{of irreducible ideals} \end{array} \right\}$

$\sum \neq \phi$ and R is Noetherian

$\Rightarrow \sum$ has a maximal element, say J.

J cannot be irreducible because, if it is, then $J = J$ written as finite intersection of irreducible ideals $\Rightarrow J \notin \sum$.

Hence $J = J_1 \cap J_2$ where $J_1 \supsetneq J$ and $J_2 \supsetneq J$.

Hence, by maximality of J; $J_1, J_2 \notin \sum$. So they can be written as $J_1 = K_1 \cap \dots \cap K_r$, $J_2 = L_1 \cap \dots \cap L_s$ finite intersection of irreducible ideals

$\Rightarrow J = (K_1 \cap \dots \cap K_r) \cap (L_1 \cap \dots \cap L_s)$ contradiction!

Prop. (45.2) $\Rightarrow$ every irreducible ideal is a primary ideal.

$\rightsquigarrow \quad I = \tilde{Q}_1 \cap \dots \cap \tilde{Q}_N \; ; \quad \tilde{Q}_1, \dots, \tilde{Q}_N$ are primary.

Using (4) on page 3 above, we may:

- First throw away irrelevant terms from $\{\tilde{Q}_1, \dots, \tilde{Q}_N\}$.
- Combine those intersections which have same <u>Radicals</u>. That is, <u>if</u>

$$I = \left(\tilde{Q}_1 \cap \dots \cap \tilde{Q}_{k_1}\right) \cap \left(\tilde{Q}_{k_1+1} \cap \dots \cap \tilde{Q}_{k_1+k_2}\right) \cap \dots \cap \underbrace{\left(\tilde{Q}_{k_1+\dots+k_{\ell-1}+1} \cap \dots \cap \tilde{Q}_{k_1+\dots+k_\ell}\right)}$$

these have same $\left[\mathrm{Rad}(\tilde{Q}_1) = \dots = \mathrm{Rad}(\tilde{Q}_{k_1})\right] \rightsquigarrow$ say P_1

same radical, say P_2 $(P_2 \neq P_1)$

...

<u>then</u> $\quad Q_1 := \tilde{Q}_1 \cap \dots \cap \tilde{Q}_{k_1}$ and so on... $Q_\ell =$

$\square$

(45.4) <u>Cor.</u> Let $\mathrm{Min}(I) := \left\{ P \subsetneq R \text{ prime ideal} \;\middle|\; I \subset P \text{ and } P \text{ is } \underline{\text{minimal}} \right\}$

$\left[\text{If } I \subset P' \subset P,\ P' \text{ another prime ideal, then } P' = P\right]$

Then $\mathrm{Min}(I) \subset \{P_1, \dots, P_\ell\}$

(Hence, $|\mathrm{Min}(I)| < \infty$).

Proof. Let $P \in \mathrm{Min}(I)$. Then $Q_1 \cap \dots \cap Q_\ell \subset P$

P : prime $\Longrightarrow$ $Q_j \subset P$ for some j (This was homework.)

$\Longrightarrow$ $\mathrm{Rad}(Q_j) = P_j \subset P$ for that j ((1) on page 2).

P minimal $\Longrightarrow$ $P_j = P$. $\square$

(45.5) Now we discuss the uniqueness of the primary ideals and their radicals, appearing in Theorem (45.3) above.

So, let $I \subsetneq R$ be a proper ideal in a Noetherian ring R. As promised by Theorem (45.3), we have primary ideals $Q_1, \dots, Q_\ell \subsetneq R$ such that

(1) $I = Q_1 \cap \dots \cap Q_\ell$ is free of redundant terms

i.e. $Q_i \not\supset \bigcap_{\substack{1 \le j \le \ell \\ j \neq i}} Q_j$

(2) $P_1 = \mathrm{Rad}(Q_1), \dots, P_\ell = \mathrm{Rad}(Q_\ell)$ are distinct.

By Cor. (45.4); for some $k \in \{1, \dots, \ell\}$ (after renaming)

$$\{P_1, \dots, P_k\} = \mathrm{Min}(I)$$

(set of minimal prime ideals containing I).

Theorem 2 (1 continued): (1) $\{P_1, \dots, P_\ell\}$ is uniquely determined by I. This set is usually called the set of prime ideals associated to I, denoted by $\mathrm{Assoc}(I)$.

$\{P_1, \dots, P_k\} = \mathrm{Min}(I)$ "minimal primes of I"	$\{P_{k+1}, \dots, P_\ell\}$ "embedded primes of I"
	More terminology

(2) $Q_1, \dots, Q_k$ are uniquely determined by I

These are often called "primary components of I".

(45.6) Proof of (1) of Thm 2 (45.5) rests on the following

Homework problem.

Lemma. Let R be a commutative ring, $Q \subsetneq R$ a primary ideal and $P = \mathrm{Rad}(Q)$. For $x \in R$, we have

(1) $x \in Q \iff (Q : x) = R$

(2) $x \notin Q \Rightarrow (Q : x)$ is primary and $\mathrm{Rad}(Q : x) = P$.

(3) $x \notin P \Rightarrow (Q : x) = Q$.

Proof. (1) is by defn. of $(Q : x) = \{r \in R \mid rx \in Q\}$.

(3): Assume $x \notin P = \mathrm{Rad}(Q)$; i.e., $x^m \notin Q$ for every $m \geq 1$.

Then $y \in (Q:x) \iff xy \in Q$ combined w/ $x^m \notin Q$ for $m \geq 1$

$\Rightarrow y \in Q$ (since Q is primary).

(2) If $x \notin Q$; then $y \in (Q:x)$
$\iff xy \in Q \;(x \notin Q) \Rightarrow y^n \in Q$ for some $n \geq 1$

Hence $(Q:x) \subset \mathrm{Rad}(Q) = P$

$\Rightarrow$ ((3) of page 3) $\mathrm{Rad}(Q:x) \subset P$.

As $Q \subset (Q:x)$, we also have $\mathrm{Rad}(Q) \subset \mathrm{Rad}(Q:x)$

$\Rightarrow \mathrm{Rad}(Q:x) = P$.

Now we show that $(Q:x)$ is primary. So, let $a.b \in (Q:x)$, $a \notin (Q:x)$

We have to prove that $b^n \in Q$ for some $n \geq 1$.

$ab \in (Q:x)$, $a \notin (Q:x)$ $\iff$ $abx \in Q$, $ax \notin Q$. As Q is primary, we get $b^n \in Q$ for some $n \geq 1$. $\square$

Returning to $I = Q_1 \cap \ldots \cap Q_\ell$ from (45.5) above -

we obtain –

$$\{P_1, \ldots, P_\ell\} = \left\{ \mathrm{Rad}(I:x) \;\middle|\; \begin{array}{c} x \in R \text{ is such} \\ \text{that } \mathrm{Rad}(I:x) \text{ is} \\ \text{prime} \end{array} \right\}$$

this set depends on $I = Q_1 \cap \ldots \cap Q_\ell$ whose uniqueness we don't know yet.

this set depends only on I

<u>Pf.</u> If $P = \mathrm{Rad}(I:x)$ for an element <u>$x \in R$ so that P is prime</u>

then
$$P = \mathrm{Rad}(Q_1 \cap \ldots \cap Q_\ell : x) = \bigcap_{i=1}^{\ell} \mathrm{Rad}(Q_i : x)$$

[Same Homework problem : $I_1 \cap \ldots \cap I_\ell = P \Rightarrow I_j = P$ for some j.]

$$\Rightarrow \mathrm{Rad}(Q_i : x) = P \qquad \Rightarrow \qquad P_i = P.$$

Conversely, write $I = Q_1 \cap \ldots \cap Q_\ell$ and let $i \in \{1, \ldots, \ell\}$.

As $Q_i \not\supseteq \bigcap_{\substack{1 \le j \le \ell \\ j \ne i}} Q_j$, we ~~get~~ have an element

$$x \in \bigcap_{\substack{1 \le j \le \ell \\ j \ne i}} Q_j \;;\; x \notin Q_i$$

Then
$$(I : x) = \Big(\bigcap_{\substack{1 \le j \le \ell \\ j \ne i}} (Q_j : x) \Big) \cap (Q_i : x)$$

all $= R$ as $x \in Q_j$ $(j \ne i)$

Some primary ideal with radical $= P_i$

Lemma 45.6 page 10 above

Hence $P_i = \mathrm{Rad}(I : x)$ for some $x \in R$. $\square$

(45.7) <u>Proof of (2) of Theorem 2 - page 10</u>:

Recall: $I = Q_1 \cap \cdots \cap Q_\ell$

$$\{P_1, \ldots, P_k\} = \mathrm{Min}(I)$$
$$\cap$$
$$\{P_1, \ldots, P_k, P_{k+1}, \ldots, P_\ell\} = \mathrm{Assoc.}(I).$$

Choose $i \in \{1, \ldots, k\}$ and let $P = P_i$; $Q = Q_i$.
(so $P = P_i \supset I$ is a minimal prime containing I.)

<u>Prop</u>: $\boxed{Q = j^*(S^{-1}I)}$ where $S = R \setminus P$ mult. closed since P is a prime ideal

$$j : \underset{\substack{\cup \\ I}}{R} \longrightarrow S^{-1}R \quad \text{ring hom}$$

Recall: $S^{-1}I = \left\{ \frac{a}{s} : a \in I, \boxed{s \in S} \right\}$ i.e. $s \notin P$

$$j^*(\tilde{I}) = \{ a \in R \mid j(a) \in \tilde{I} \}$$
(for any ideal $\tilde{I} \subset S^{-1}R$; e.g. $\tilde{I} = S^{-1}I$!)

Also recall (Lecture 41 - page 3)

$$j^*(S^{-1}I) = \{ a \in R \mid ta \in I \text{ for some } t \in S \}$$

<u>Proof</u>. Given a primary ideal $Q' \in \{Q_1, \ldots, Q_\ell\}$
$Q' \neq Q$, we know $Q' \not\subset P$. This is because

if $Q' \subset P$ then $\underset{\substack{\uparrow \\ \text{a prime ideal} \\ \neq P, \text{ from} \\ \{P_1, \ldots, P_\ell\}}}{\mathrm{Rad}(Q')} \subset P$ contradicting minimality of P.

$$\Rightarrow \quad Q' \cap S \neq \phi \quad \Rightarrow \quad S^{-1}Q' = S^{-1}R.$$

Hence $I = Q_1 \cap \ldots \cap Q_\ell \;\Rightarrow\; S^{-1}I \underset{\text{homework!}}{=} (S^{-1}Q_1) \cap \ldots \cap (S^{-1}Q_\ell)$

$$= S^{-1}Q \quad \left(\begin{array}{l}\text{the rest are} \\ \text{all } = S^{-1}R\end{array}\right)$$

$$\Rightarrow j^*(S^{-1}I) = j^*(S^{-1}Q) \underset{\text{always!}}{\supset} Q$$

Conversely, $a \in j^*(S^{-1}Q) \Rightarrow \exists\, t \in S$ s.t. $t \cdot a \in Q$.

$t \in S$ means $t \notin P = \mathrm{Rad}(Q)$ means $t^n \notin Q$ for any $n \geq 1$.

As $t \cdot a \in Q$, $t^n \notin Q$ $(\forall n \geq 1)$, Q is primary, we get $a \in Q$. $\square$

(45.8) <u>An example</u> (also from homework set - optional one!)

$$\underset{\substack{\uparrow \\ \text{a field}}}{R = K[x, y]} \supsetneq I = (x^2, xy) = \{f_1(x,y) \cdot x^2 + f_2(x,y) \cdot xy \mid f_1, f_2 \in R\}$$

- $\underline{\mathrm{Min}(I) = \{P_1 := (x)\}}$:

$(x) \subset K[x,y]$ is a prime ideal because $K[x,y]/(x) \cong K[y]$ is a domain. $I \subset P'$ any prime ideal
$\Rightarrow x^2 \in P' \Rightarrow x \in P' \Rightarrow P_1 := (x) \subset P'$.

- For each $n \geq 2$; $I_n := (x^2, xy, y^n) \supset I$ is primary [Ex.].

(Hint: $f \in K[x,y]/(x^2, xy, y^n) \Rightarrow f$ is a unit or nilpotent.)

- $I = (x) \cap I_n \quad (\forall\, n \geq 2) \quad \left(\begin{array}{c} I \subset (x) \\ \text{and } I \subset I_n \end{array}\right)$.

[$\mathrm{Rad}(I_n) = (x, y) =: P_2$ "embedded prime"
— proof - easy exercise!]

Proof: $\underline{\text{RHS} \subset \text{LHS}}$:

$f \in (x) \Rightarrow f = x \cdot g$ for some $g \in R$

$f \in (x^2, xy, y^n) \Rightarrow f = f_1 x^2 + f_2 xy + f_3 \cdot y^n$

set $x = 0$

$\left(f(x,y) = x \cdot g(x,y) \Big|_{x=0} = \right) 0 = f_3(0,y) \cdot y^n$

i.e. $f_3(0,y) = 0 \Rightarrow f_3$ is divisible by $x \Rightarrow$

$f = f_1 x^2 + \cancel{f_2}\, xy \left(f_2 + \frac{f_3}{x} y^{n-1} \right) \in (x^2, xy)$ ✓ □

Pictorial interpretation.

Ideals		Pictures in $\mathbb{R}^2$
$I \subset \mathbb{R}[x,y]$	$\rightsquigarrow$ $\not\leftarrow$	$\{(a,b) \in \mathbb{R}^2 \mid f(a,b) = 0 \ \forall f \in I\}$

$(x^2, xy) \quad \rightsquigarrow \quad \{(0,b) \in \mathbb{R}^2 \mid b \in \mathbb{R}\} = X$

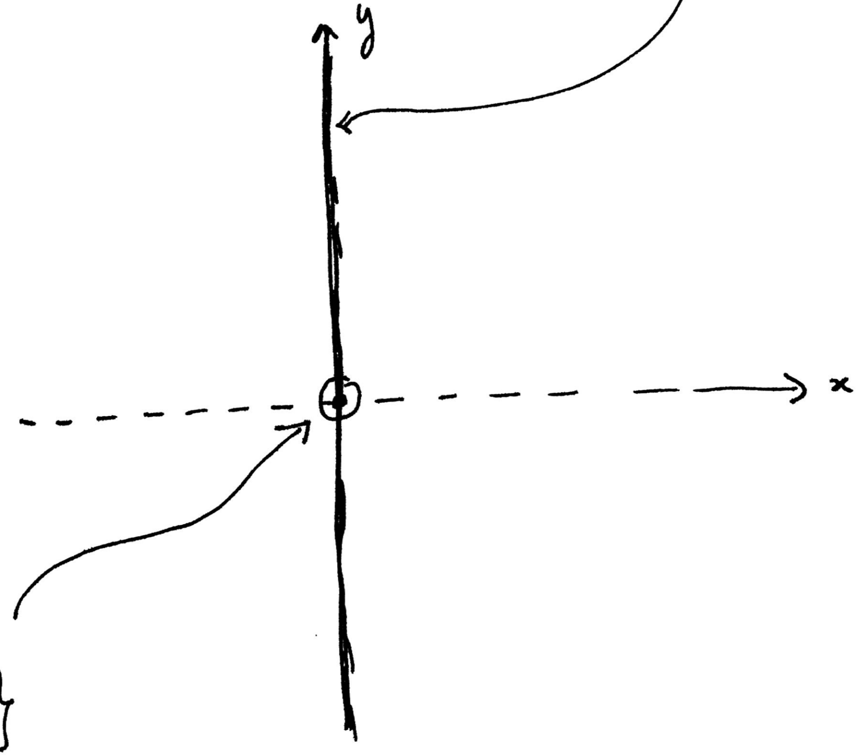
"(0,0) is there twice"

but; also

$\underline{\underline{(x)}} \quad \rightsquigarrow \quad \{(0,b) \in \mathbb{R}^2 \mid b \in \mathbb{R}\}$

$\boxed{\text{Minimal Prime}}$

X: the set is "hiding" (0,0) behind!



$(x,y) \quad \rightsquigarrow \quad \{(0,0)\}$

$\cap$ ideal

$\mathbb{R}[x,y]$

$\boxed{\text{Embedded Prime}}$