HOMEWORK 1

Problem 1. Let $\sum_{k=0}^{\infty} a_k z^k \in \mathbb{C}[[z]]$ be a power series with radius of convergence 1. Let $\{y_n\}_{n\geq 0}$ be defined by:

$$y_0 = 1, \qquad \text{and} \qquad y_n = \frac{1}{n}\sum_{j=0}^{n-1} a_j y_{n-1-j}, \forall n \geq 1.$$

Prove that the radius of convergence of $\sum_{k=0}^{\infty} y_k z^k$ is at least 1.

Problem 2. Recall the definition of the hypergeometric series:

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{(a)_n (b)_n}{(c)_n},$$

where we use the notation $(p)_n = p(p+1)\cdots(p+n-1)$ if $n \geq 1$, and $(p)_0 = 1$.

(i) Prove that $\frac{d}{dz}F(a, b; c; z) = \frac{ab}{c}F(a+1, b+1; c+1; z)$.

(ii) Prove that $F(a, b; c; z)$ is a solution of the following second order linear ODE (called the hypergeometric equation).

$$z(1-z)u'' + (c - (a+b+1)z)u' - abu = 0.$$

(iii) Prove that $(1-z)^{c-a-b}F(c-a, c-b; c; z)$ also solves the hypergeometric equation, and takes value 1 at $z = 0$. Use this to conclude that $F(a, b; c; z) = (1-z)^{c-a-b}F(c-a, c-b; c; z)$.

Problem 3. Verify the solution given in Lecture 2, §3, page 5.

(Very Optional) Compute the matrix K relating the solutions near $z = 0$ and $z = 1$ (see Lecture 1, page 8) for the differential equation given in Lecture 2, §3.

Problem 4. Let $z^{-1}\mathbb{C}[[z^{-1}]]$ be the ring of formal series in z^{-1} with zero constant term. Consider the (formal) Borel transform: $z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[p]]$ given by:

$$\mathcal{B} : \sum_{n\geq 0} c_n z^{-n-1} \mapsto \sum_{n\geq 0} c_n \frac{p^n}{n!}$$

which we will denote by $F(z) \mapsto \mathcal{B}(F)(p)$.

(1) Prove that $\mathcal{B}(-\partial_z F)(p) = p\mathcal{B}(F)(p)$.

(2) Let $c \in \mathbb{C}$ and $F(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$. Consider the shift operator $(T_c F)(z) := F(z-c)$. Prove that $\mathcal{B}(T_c F)(p) = e^{cp}\mathcal{B}(F)(p)$.

(3) Compute $\mathcal{B}(F)$ for the following $F(z)$:

$$F(z) = \frac{1}{(z-c)^{\ell}}, \ell \in \mathbb{Z}_{\geq 1}; \qquad F(z) = \sum_{n=0}^{\infty} n! z^{-n-1}.$$