HOMEWORK 2

Problem 1. Let $n \geq 2$ and consider the following system of PDE's:

$$\frac{\partial f}{\partial z_i} = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{t_{ij} \cdot f}{z_i - z_j}, \quad \text{for every } 1 \leq i \leq n.$$

where $f(z_1, \ldots, z_n)$ takes values in a finite–dimensional vector space F over $\mathbb{C}$, and $t_{ij} = t_{ji} \in \operatorname{End} F$.

(1) Prove that (using Kohno's lemma) this system is consistent if, and only if
  - For (i,j) and (k,l) distinct, we have $[t_{ij}, t_{kl}] = 0$.
  - For a triple (i,j,k) of distinct indices, we have $[t_{ij}, t_{jk} + t_{ik}] = 0$.

(2) Assume that we have a representation of S_n (symmetric group) on F (that is, we are given a group homomorphism $\rho : S_n \to \mathrm{GL}(F)$). Verify the relations from the previous part, when we set, for $i \neq j$, $t_{ij} = \rho(s_{ij})$. Here $s_{ij} \in S_n$ is the transposition $i \leftrightarrow j$.

Problem 2. Let $R \subset E^* \setminus \{0\}$ be a (finite) root system, and let W be its Weyl group. We assume that a fundamental chamber $\mathcal{C}^0$ has been chosen. Prove that, for $w \in W$, we have: $\ell(w) = \#\{\alpha \in R_+ | w(\alpha) \in R_-\}$.

Problem 3. Prove that W has a unique element of maximum length. What is it for $W = S_n$?

Problem 4. Let $\{\alpha_i\}_{i \in I}$ denote the set of simple roots (fundamental chamber is chosen). For $\alpha \in R_+$, define height of α as:

$$\alpha = \sum_{i \in I} n_i \alpha_i \Rightarrow \mathrm{ht}(\alpha) := \sum_{i \in I} n_i.$$

Prove that R_+ has a unique element of maximum height. What is it for A_n?

Problem 5. Define:

$$\mathrm{Aut}(R) := \{f : E^* \to E^* \text{ linear such that } (f(\alpha), f(\beta)) = (\alpha, \beta) \text{ and } f(R) = R\}$$

Prove that W is a normal subgroup of $\mathrm{Aut}(R)$. Prove that the quotient $\mathrm{Aut}(R)/W$ is naturally identified with the symmetries of the Dynkin diagram of R.