HOMEWORK 3

Problem 1. Let $\mathfrak{g}$ be the simple Lie algebra associated to the root system B_2. Recall that the Cartan matrix of B_2 is $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$. Express the root vectors $\{e_\alpha\}_{\alpha\in R_+}$ as commutators of $\{e_1, e_2\}$.

Problem 2. (We continue with the root system of type B_2). Let F be a finite–dimensional vector space over $\mathbb{C}$ and let $\{t_1, t_2, t_3, t_4\} \subset \mathrm{End}(F)$ be such that $t_1 + t_2 + t_3 + t_4$ is a scalar matrix (hence commutes with each t_j). Consider the following connection:

$$\nabla = d - \left(\frac{d\alpha_1}{\alpha_1} t_1 + \frac{d\alpha_2}{\alpha_2} t_2 + \frac{d(\alpha_1+\alpha_2)}{\alpha_1+\alpha_2} t_3 + \frac{d(2\alpha_1+\alpha_2)}{2\alpha_1+\alpha_2} t_4 \right).$$

There are two maximal nested sets: $\mathcal{F} = \{\{\alpha_1\}, R_+\}$ and $\mathcal{G} = \{\{\alpha_2\}, R_+\}$. Verify that the De Concini–Procesi associator $\Phi_{\mathcal{G},\mathcal{F}}$ is same as the associator relating fundamental solutions near 0 and 1 of the following ODE (see Lecture 12 §3, page 3):

$$\frac{df}{du} = \left(\frac{t_1}{u} + \frac{t_2}{u-1} + \frac{t_3}{u+1} \right) f.$$

Problem 3. Let $\mathfrak{g} = \mathfrak{sl}_2$ and let L_n be the $n+1$–dimensional, irreducible representation, with basis denoted by $\{v_i^{(n)}\}_{0\le i\le n}$ (see Lecture 14 §0, page 1). Consider the representation $V = L_m \otimes L_n$, where $m, n \in \mathbb{Z}_{\ge 0}$ and we assume $m \ge n$.

(1) What is the dimension of the weight space $V[m+n-2r]$, where $0 \le r \le m+n$?

(2) Let $0 \le r \le n$, and consider the following vector in $V[m+n-2r]$: $\xi(r) = \sum_{j=0}^{r} c_j v_j^{(m)} \otimes v_{r-j}^{(n)}$. Assuming $c_0 = 1$, compute c_j $(1 \le j \le r)$ such that $e \cdot \xi(r) = 0$.

(3) Use the previous parts to prove that

$$L_m \otimes L_n \cong L_{m+n} \oplus L_{m+n-2} \oplus \cdots \oplus L_{m-n}.$$