HOMEWORK 4

Notations

Let $q \in \mathbb{C}$ be such that $|q| > 1$. Recall our notations for q–numbers:

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad \left[\begin{array}{c} m \\ n \end{array}\right] := \frac{[m][m-1]\cdots[m-n+1]}{[n]!}$$

The problems below involve the quantum group $U_q(\mathfrak{sl}_2)$. Its generators are $\{K^{\pm 1}, E, F\}$ subject to the following list of relations:

$$KE = q^2 EK \qquad KF = q^{-2} FK$$

$$EF = FE + \frac{K - K^{-1}}{q - q^{-1}}$$

The coproduct $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ is given by: $\Delta(K) = K \otimes K$, and

$$\Delta(E) = E \otimes 1 + K \otimes E,$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F.$$

For $\lambda \in \mathbb{C}$, let M_λ be the Verma module of $U_q(\mathfrak{sl}_2)$: it has basis $\{m_\lambda(r) : r \geq 0\}$ with the action of $\{K, E, F\}$ given by: $K \cdot m_\lambda(r) = q^{\lambda - 2r} m_\lambda(r)$, and:

$$E \cdot m_\lambda(r) = [\lambda - r + 1] m_\lambda(r-1), \quad F \cdot m_\lambda(r) = [r+1] m_\lambda(r+1).$$

For $n \in \mathbb{Z}_{\geq 0}$, let L_n be the $(n+1)$–dimensional irreducible representation of $U_q(\mathfrak{sl}_2)$. It has a basis $\{v_k^{(n)} : 0 \leq k \leq n\}$ with $U_q(\mathfrak{sl}_2)$ action given by the same formulae as above.

Problems

Problem 1. Let x, y be two non–commuting variables such that $xy = q^2 yx$. Prove that, for every $n \geq 1$, we have:

$$(x+y)^n = \sum_{\ell=0}^{n} q^{\ell(n-\ell)} \left[\begin{array}{c} n \\ \ell \end{array}\right] y^\ell x^{n-\ell}.$$

Problem 2. Recall that we defined q–exponential as:

$$\exp_q(x) := \sum_{n \geq 0} q^{n(n-1)/2} \frac{x^n}{[n]!}.$$

Prove that $\exp_q(x)^{-1} = \exp_{q^{-1}}(-x)$. (*Hint: verify that both* $\exp_q(x)$ *and* $\exp_{q^{-1}}(-x)^{-1}$ *satisfy the same functional equation:* $f(qx) - f(q^{-1}x) = (q - q^{-1}) x f(qx)$*, with the normalization condition* $f(0) = 1$.)

Problem 3. Prove the following commutation relation in $U_q(\mathfrak{sl}_2)$, for every $n \geq 1$. Recall our notation: $F^{(k)} = \frac{F^k}{[k]!}$.

$$EF^{(n)} = F^{(n)}E + \frac{q^{n-1}K - q^{-n+1}K^{-1}}{q - q^{-1}} F^{(n-1)}.$$

Problem 4. Let $\mu \in \mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq n$. Consider the following vector $\xi_k^{(n)}(\mu) \in M_\mu \otimes L_n$ of weight $\mu + n - 2k$:

$$\xi_k^{(n)}(\mu) := \sum_{\ell \geq 0} (-1)^\ell q^{\ell(\mu - \ell + 1)} \frac{\begin{bmatrix} n-k+\ell \\ \ell \end{bmatrix}}{\begin{bmatrix} \mu \\ \ell \end{bmatrix}} m_\mu(\ell) \otimes v_{k-\ell}^{(n)}.$$

Verify that $E \cdot \xi_k^{(n)}(\mu) = 0$.

Thus, for $\lambda \in \mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq n$, we have a unique intertwiner

$$\varphi_k^{(n)}(\lambda) : M_\lambda \to M_\mu \otimes L_n,$$

where $\mu = \lambda - n + 2k$, given by: $\varphi_k^{(n)}(\lambda) : m_\lambda(0) \mapsto \xi_k^{(n)}(\mu)$.

Problem 5. Now let $\lambda \in \mathbb{C}$, $n_1, n_2 \in \mathbb{Z}_{\geq 0}$ and $0 \leq k_j \leq n_j$ $(j = 1, 2)$. Recall that, by definition, the fusion operator $J_{L_{n_1}, L_{n_2}}(\lambda)$ evaluated on $v_{k_1}^{(n_1)} \otimes v_{k_2}^{(n_2)}$, is obtained from the following composition: (here $\mu_2 = \lambda - (n_2 - 2k_2)$ and $\mu = \mu_2 - (n_1 - 2k_1)$):

$$\psi := \left(\varphi_{k_1}^{(n_1)}(\mu_2) \otimes \mathrm{Id}_{L_{n_2}}\right) \circ \varphi_{k_2}^{(n_2)}(\lambda) \ : \ M_\lambda \to M_{\mu_2} \otimes L_{n_2} \to M_\mu \otimes L_{n_1} \otimes L_{n_2}.$$

That is:

$$\psi(m_\lambda(0)) = m_\mu(0) \otimes \left(J_{L_{n_1}, L_{n_2}}(\lambda) \cdot (v_{k_1}^{(n_1)} \otimes v_{k_2}^{(n_2)})\right) + \dots$$

Prove that:

$$J_{L_{n_1}, L_{n_2}}(\lambda) \cdot \left(v_{k_1}^{(n_1)} \otimes v_{k_2}^{(n_2)}\right) =$$

$$\sum_{\ell \geq 0} (-1)^\ell q^{\ell(\mu_2 - \ell + 1)} \frac{\begin{bmatrix} n_2 - k_2 + \ell \\ \ell \end{bmatrix} \begin{bmatrix} k_1 + \ell \\ \ell \end{bmatrix}}{\begin{bmatrix} \mu_2 \\ \ell \end{bmatrix}} v_{k_1+\ell}^{(n_1)} \otimes v_{k_2-\ell}^{(n_2)}.$$

Problem 6. Using the last problem, show that the fusion operator has the following limit (once both sides are evaluated on $L_{n_1} \otimes L_{n_2}$):

$$\lim_{q^\lambda \to \infty} J(\lambda) = \exp_{q^{-1}}(-(q - q^{-1}) F \otimes E).$$

Note that the right–hand side here is the inverse of $\overline{\mathcal{R}} = \exp_q((q - q^{-1}) F \otimes E)$ by Problem 2 above.