Lecture 1.

Ordinary differential equations in the complex plane

0. Set up. - Let $D \subset \mathbb{C}$ be an open, connected set; and $n \geq 1$.

We are going to study equation of the form

$$(*) \quad — \quad \boxed{\frac{dF}{dz} = A(z)\,F(z)}$$

where $A : D \longrightarrow M_n(\mathbb{C})$ ($n\times n$ matrices with entries from $\mathbb{C}$); is a given (holomorphic / meromorphic) function; and F is to be determined. We will look for $F : D \longrightarrow GL_n(\mathbb{C})$ (i.e., F takes values in $M_n(\mathbb{C})$ and $\det F(z) \not\equiv 0$).

1. Some easy facts about matrix-valued functions.:

(a) $$\frac{d}{dz}\left(F(z)\cdot G(z)\right) = F'(z)\,G(z) + F(z)\,G'(z).$$

(b) $$\frac{d}{dz}\left(F(z)^{-1}\right) = -F(z)^{-1}\left(\frac{d}{dz}F(z)\right)F(z)^{-1}$$

($F(z)^{-1}$ = matrix inverse.)

[Proof. Let $G(z) = F(z)^{-1}$. Then $0 = \frac{d}{dz}\left(F(z)\,G(z)\right)$

$= F'(z)\,G(z) + F(z)\,G'(z) \Rightarrow G'(z) = -F(z)^{-1}F'(z)\,F(z)^{-1}$.]

(c) If F_1 and F_2 are two ($GL_n(\mathbb{C})$-valued) solutions of $(*)$; then there exists $C \in GL_n(\mathbb{C})$ such that

$$F_2(z) = F_1(z) \cdot C.$$

[Proof : $$\frac{d}{dz}\left(F_1^{-1} F_2\right) = -F_1^{-1} F_1' F_1^{-1} F_2 + F_1^{-1} F_2'$$
$$= -F_1^{-1} A \cancel{F_1} \cancel{F_1^{-1}} F_2 + F_1^{-1} A F_2$$
$$= 0. \quad]$$

2. Solution near an ordinary point.

For simplicity, assume that D is a disc around 0, and $A(z)$ is <u>holomorphic near 0</u>. In this case, we say $0 \in D$ is an ordinary point of our equation $F'(z) = A(z) F(z)$.

Prop. There exists a unique solution of $F'(z) = A(z) F(z)$, holomorphic near 0, normalized as $F(0) = Id_{n\times n}$ (or just 1 for convenience).

Proof. Consider the Taylor series of $A(z)$ near 0:
$$A(z) = A_0 + A_1 z + A_2 z^2 + \cdots$$
where $A_0, A_1, \ldots \in M_n(\mathbb{C})$.

For unknown $\{F_k\}_{k\geq 0}$; $F(z) = \sum_{k\geq 0} F_k z^k$ solves

$$F'(z) = A(z) F(z) \iff n F_n = \sum_{i=0}^{n-1} A_i F_{n-1-i} \quad \forall n \geq 1.$$

(compare coefficients of z^{n-1} ; $n \geq 1$)

[This argument is due to Frobenius.]

Together with $F_0 = 1$, we obtain existence and uniqueness of a formal solution. The question of convergence of $\sum_{k \geq 0} F_k z^k$ is usually handled in the following form:

if $\sum_{n \geq 0} a_n z^n$ ($a_n \in \mathbb{C}$ $\forall n = 0, 1, \ldots$) has radius of convergence $r > 0$, then $\{f_k\}_{k \geq 0}$ defined by $\left\{ \begin{array}{l} f_0 = 1 \\ f_n = \dfrac{\sum_{i=0}^{n-1} a_i f_{n-1-i}}{n} \\ (n \geq 1) \end{array} \right\}$ give rise to the power series $\sum_{k \geq 0} f_k z^k$ of radius of convergence $= r$.

(The proof of this fact is ~~an~~ an easy exercise.) $\square$

3. Solutions near a regular singular (also called Fuchsian) point.

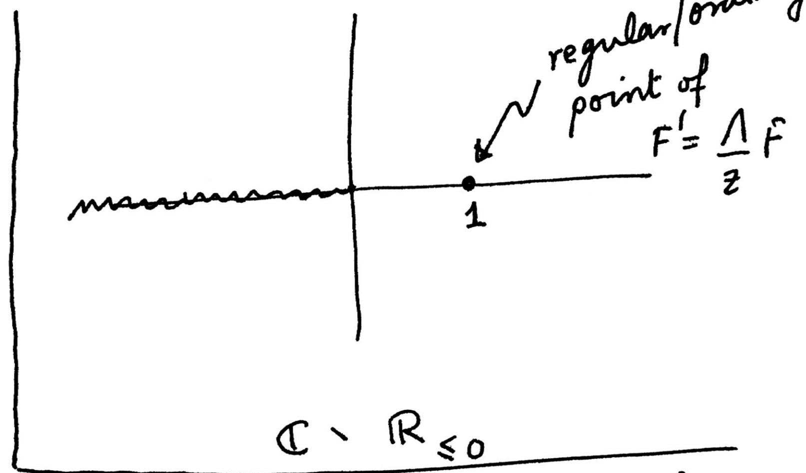
With the same assumptions on D, let us assume $A(z)$ has a pole of order 1 at 0. In this case, we say 0 is a <u>regular singular</u>, or <u>Fuchsian</u> point of $F'(z) = A(z) F(z)$.

Example. $F'(z) = \dfrac{\Lambda}{z} F(z)$ where $\Lambda \in M_n(\mathbb{C})$ is a fixed matrix. A solution of this equation is given by

$$F(z) = z^{\Lambda} = \exp(\Lambda \ln(z)).$$

$\ln(z)$ is defined to be the principal branch of logarithm:

$$\ln : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C} \qquad \text{value} = 0 \text{ at } z = 1.$$



$F(z) = z^{\Lambda}$ is the unique solution taking value 1 at $z = 1$.

If γ is a loop around 0, say, $\gamma : [0,1] \to \mathbb{C}$, $t \mapsto e^{2\pi i t}$, then continuation of $F(z)$ along γ gives $\tilde{F}(z) = z^{\Lambda} \cdot \exp(2\pi i \Lambda)$ (another solution near 1.)

$$\mu(\gamma) \overset{\text{def}}{=} F(z)^{-1} \tilde{F}(z) = \exp(2\pi i \Lambda).$$

(with respect to $F(z)$, near $z = 1$.)

4. Fuchsian point continued. $F'(z) = A(z) F(z)$ where

$$A(z) = \frac{\Lambda}{z} + A_0 + A_1 z + A_2 z^2 + \cdots$$

$$A : D \setminus \{0\} \to M_n(\mathbb{C}).$$

Let $H(z) = F(z) z^{-\Lambda}$ (i.e. $F(z) = H(z) \cdot z^{\Lambda}$). Then $H(z)$ satisfies the following equation:

$$H'(z) = \frac{[\Lambda, H(z)]}{z} + \left(\sum_{k\geq 0} A_k z^k\right) . H(z) \quad ; \quad \text{where}$$

$$[A_1, A_2] := A_1 A_2 - A_2 A_1 .$$

[Proof. - $F(z) = H(z) . z^{\Lambda}$ solves $F' = AF$

$$\Longleftrightarrow \quad H'(z). z^{\Lambda} + H(z) . \frac{\Lambda}{z} z^{\Lambda} = \left(\frac{\Lambda}{z} + \sum_{k\geq 0} A_k z^k\right) H(z) . z^{\Lambda}$$

Cancelling z^{Λ} on the right gives the equation we claimed.]

Again, writing formally, $H(z) = \sum_{m\geq 0} H_m z^m$ and normalizing $H_0 = 1$; we get

$$H'(z) = \frac{[\Lambda, H(z)]}{z} + \left(\sum_{k\geq 0} A_k z^k\right)\left(\sum_{\ell\geq 0} H_\ell z^\ell\right) \rightsquigarrow \overset{\text{coeff of } z^{-1}:}{0 = [\Lambda, H_0]} \checkmark$$

and $\forall\, m \geq 0$: $\quad (m+1) H_{m+1} = [\Lambda, H_{m+1}] + \sum_{k=0}^{m} A_k H_{m-k}$

(Coeff. of z^m on both sides)

<u>Define</u> $\quad \mathrm{ad}(\Lambda) : \mathcal{M}_n(\mathbb{C}) \longrightarrow \mathcal{M}_n(\mathbb{C})$

$$X \longmapsto [\Lambda, X]$$

$$\left(\text{eg. } \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \Rightarrow \mathrm{ad}(\Lambda)(X) = \left((\lambda_i - \lambda_j) X_{ij}\right)_{1\leq i,j\leq n}\right)$$

We can write the recurrence relation for $\{H_m\}_{m\geq 0}$ as :

$$\left(m+1-\mathrm{ad}(\Lambda)\right)\cdot H_{m+1} = \sum_{k=0}^{m} A_k H_{m-k} \quad (\forall\, m \geq 0).$$

We say Λ is <u>non-resonant</u> if <u>eigenvalues of Λ do not differ by non-zero integers</u>. This assumption makes sure that $m+1-\mathrm{ad}(\Lambda) : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ is invertible $\forall\, m \geq 0$; and hence $H(z)$ is uniquely determined:

$$H_0 = 1. \qquad \underset{(m\geq 0)}{H_{m+1}} = (m+1-\mathrm{ad}\,\Lambda)^{-1}\cdot\left(\sum_{k=0}^{m} A_k H_{m-k}\right).$$

We have thus proved:

<u>Theorem.</u> - Assuming Λ is non-resonant, the differential equation

$$\frac{dF}{dz} = A(z)F(z) \quad ; \text{ where } A(z) = \frac{\Lambda}{z} + \underbrace{A^{reg}(z)}_{\text{holomorphic near } 0.}$$

has a unique solution of the form $H(z)\cdot z^{\Lambda}$, where $H(z)$ is holomorphic near 0 and $H(0) = 1$.

[Convergence of $H(z)$ is obtained exactly as in §2 above, combined with the fact that we can find $K \in \mathbb{R}_{>0}$ such that

$$\|(m-\mathrm{ad}\,\Lambda)^{-1}\| < \frac{K}{m}.\ ]$$

Exactly as in §3, Example above

$$\mu\left(\circlearrowleft \right) = \exp\left(2\pi i \Lambda\right).$$

(w.r.t. $F(z) = H(z)\, z^{\Lambda}$)

5. **Associator.** Consider the following differential equation, where A and B are constant matrices. (assume non-resonant).

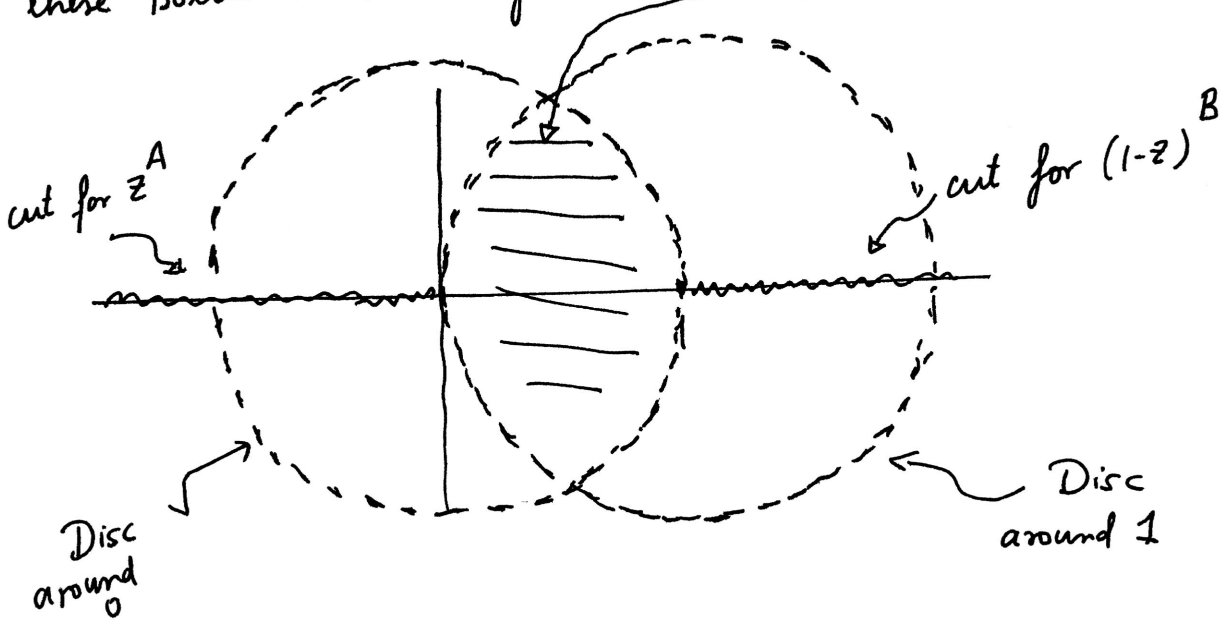
$$\boxed{\frac{dF}{dz} = \left(\frac{A}{z} + \frac{B}{z-1} \right) F(z)}$$

Using Theorem (§4) from the last page, we can find two solutions

$$F^{(0)}(z) = H^{(0)}(z) \cdot z^{A} \qquad \left(H^{(0)}(z=0) = 1 \right)$$

$$F^{(1)}(z) = H^{(1)}(z)\,(1-z)^{B} \qquad \left(H^{(1)}(z=1) = 1 \right)$$

Both of these solutions are defined on $\{|z|<1\} \cap \{|z-1|<1\}$

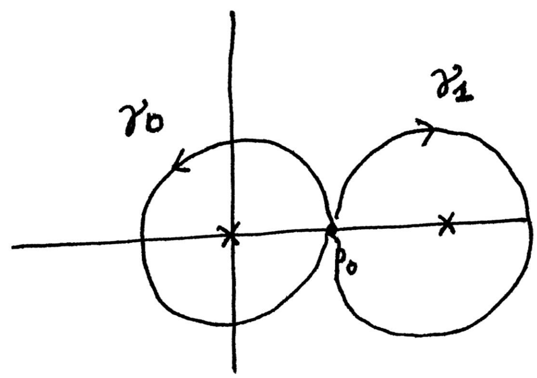


Hence there must exist $K \in GL_n(\mathbb{C})$ such that

$$F^{(0)}(z) = F^{(1)}(z) \cdot K \qquad (\forall z \in \{|w|<1\} \cap \{|w-1|<1\})$$

We say K is the associator for [Drinfeld] $\dfrac{dF}{dz} = \left(\dfrac{A}{z} + \dfrac{B}{z-1}\right) F$.

If we know the Drinfeld associator, we can write the monodromy explicitly:



$$\mu_{F^{(0)}}(\gamma_0) = \exp(2\pi i A)$$

$$\mu_{F^{(1)}}(\gamma_1) = \exp(2\pi i B)$$

Check. $\mu_{F(z)\cdot C}(\gamma) = C^{-1} \mu_{F(z)}(\gamma)\, C$

Hence $\mu_{F^{(0)}}(\gamma_1) = \mu_{F^{(1)}\cdot K}(\gamma_1) = K^{-1} \mu_{F^{(1)}}(\gamma_1)\, K$

$$= K^{-1} \exp(2\pi i B)\, K.$$