Lecture 10

0. Summary so far. Recall (from Lecture 4) that we introduced the following rational connection and obtained a criterion for its flat-ness (or consistency)

$$\nabla = d - \sum_{x \in X} \frac{dx}{x} t_x$$

In the set up of lecture 4; we fixed V = a finite-dimensional $\mathbb{C}$-vector space; $X \subset V^* \setminus \{0\}$ a finite set of linear forms. We proved

Kohno's Lemma: ∇ is flat (i.e. defines a consistent system of PDE's) $\Leftrightarrow$ for every subset $Y \subset X$ maximal s.t. Span Y is 2-dim'l we have $\left[\sum_{y \in Y} t_y , t_z \right] = 0.$

Now we are going to assume that the underlying hyperplane arrangement (i.e. $X \subset V^*$ or $\{H_x = \mathrm{Ker}(x)\}_{x \in X}$ - hyperplanes in V) comes from a root system (see Lecture 5, §3).

1. Complexification. Let $\underline{E, R \subset E^* \setminus \{0\}}$ be a finite root system.

Set $\mathfrak{h} = E \otimes_{\mathbb{R}} \mathbb{C}$ (i.e. extend scalars from $\mathbb{R}$ to $\mathbb{C}$).

Each $\alpha : E \to \mathbb{R}$ is then extended to a $\mathbb{C}$-linear form

The role of "$X \subset V^* \setminus \{0\}$" will be played by R_+ = set of positive roots. Thus our rational connection will take the following form:
$$\nabla = d - \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} \cdot t_\alpha$$

For concrete-ness, let us fix another f.d. $\mathbb{C}$-vector space, say F, and assume that $t_\alpha \in \mathrm{End}(F)$ $\forall \alpha \in R_+$.

As $\dfrac{d(-\alpha)}{(-\alpha)} = \dfrac{d\alpha}{\alpha}$, we are going to consider $t_\alpha \in \mathrm{End}\, F$ for all roots $\alpha \in R$ and impose $t_\alpha = t_{-\alpha}$.

2. W-action. Let us assume that we are given, in addition, an action of W on F. That is, a group homomorphism $W \xrightarrow{a} GL(F)$. This allows us to conjugate elements of $\mathrm{End}\, F$ by $w \in W$ as: for $w \in W$, $t \in \mathrm{End}(F)$ $\quad w \cdot t := a(w)\, t\, a(w)^{-1} \in \mathrm{End}\, F$

We say <u>the connection ∇ is W-equivariant</u> if
$$w \cdot t_\alpha = t_{w(\alpha)} \quad \forall\, w \in W \text{ and } \alpha \in R.$$

3. A W-equivariant connection $\nabla = d - \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} t_\alpha$

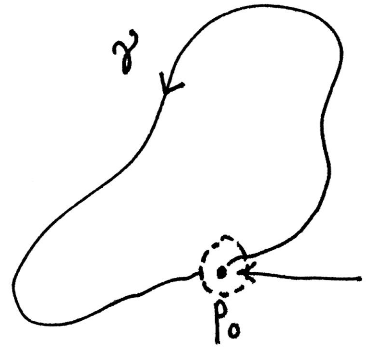
allows us to define <u>the monodromy representation</u>, which is a group homomorphism $\mu : \pi_1(\mathfrak{h}^{reg}/W) \longrightarrow GL(F)$.

[Notation : $\mathfrak{h}^{reg} = \mathfrak{h} \setminus \bigcup_{\alpha \in R_+} H_\alpha$ "regular elements of $\mathfrak{h}$".]

μ in fact depends on

- choice of a base point in $\mathfrak{h}^{reg}/W$; say p_0. (defn. of π_1 involves choosing a base point)
- a $GL(F)$-valued solution ψ of $\nabla\psi = 0$; near p_0.

4. Definition of $\mu_{(\psi, p_0)}$: Given a loop $\gamma : [0,1] \to \mathfrak{h}^{reg}/W$ consider the analytic continuation of ψ along γ



two solutions ψ and $\tilde{\psi}$ // continuation of ψ along γ.

$$\mu(\gamma) := \psi^{-1} \cdot \tilde{\psi} \in GL(F)$$

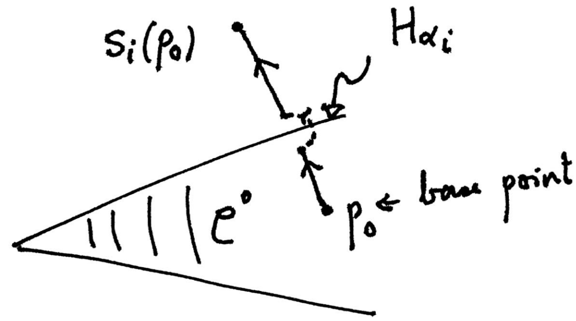
(constant since both ψ & $\tilde{\psi}$ solve the same equation).

5. The fundamental group $\pi_1(\mathfrak{h}^{reg}/W)$ was computed by Brieskorn (DIE FUNDAMENTALGRUPPE DES RAUMES DER REGULÄREN ORBITS EINER ENDLICHEN KOMPLEXEN SPIEGELUNGSGRUPPE – Invent Math 12 (57-61). 1971).

Theorem. $\pi_1(\mathfrak{h}^{reg}/W, p_0) = B_W$ (braid group of W)
(see Lecture 9, §4, page 7).

The base point is assumed to be in the fundamental chamber (i.e. $p_0 \in \mathfrak{h}^{reg}$ is such that $\alpha_i(p_0) \in \mathbb{R}_{>0}$.) For each $i \in I$, the generator T_i of B_W is then mapped to the following loop:

γ_i joins p_0 to $s_i(p_0)$ by approaching the wall H_{α_i} in the real vector space E; and circling around it in a counterclockwise direction – in the imaginary coordinate.



Picture of the loop γ_i $(i \in I)$.

6. Brieskorn's Theorem was reproved and extended ($\pi_k(\mathfrak{h}^{reg}/W) = \{1\}$ $\forall\, k \geq 2$) by Deligne - Les immeubles des groupes de tresses généralisés (Invent. Math. 1972).

7. Good / fundamental solutions.

[De Concini, Procesi - Hyperplane arrangements and holonomy equations Selecta Math. (1995).]

Now we will try to find solutions ψ of $\nabla\psi = 0$ for which the monodromy is "easy to compute". This should be viewed as several variable generalization of (Lecture 1 §4):

$$F'(z) = \left(\frac{\Lambda}{z} + A_{reg}(z)\right)F \quad \rightsquigarrow \text{fundamental solution } H(z).z^{\Lambda} = \psi$$

$$\rightsquigarrow \mu_{\psi}\left(\circlearrowleft\right) = e^{2\pi i \Lambda}$$

The problem is that of "normal crossing" (i.e., we usually have too many hyperplanes passing through $0 \in \mathfrak{h}$. (more than $\dim_{\mathbb{C}}\mathfrak{h}$)). To make this point clear, let us ~~prov~~ ~~that~~ consider the "normal crossing" case first

8. Let $n \geq 2$ and assume we have the following system of PDE's:

$$\frac{\partial f}{\partial x_i} = \left(\frac{t_i}{x_i} + R_i(\underline{x}) \right) f \qquad (1 \leq i \leq n). \quad -(*)$$

Here $f(x_1, \dots, x_n)$ is (unknown) function of n variables; taking values in $GL(F)$; $t_1, \dots, t_n \in End(F)$; and $R_i(\underline{x})$ are $End(F)$-valued, holomorphic functions near $\underline{x} = \underline{0}$.

Exercise: $(*)$ is consistent $\Leftrightarrow$

$$[t_i, t_j] = 0 \quad \forall\, i,j$$

$$[t_i, R_j] = 0 \quad \forall\, i \neq j$$

$$\frac{\partial R_i}{\partial x_j} - \frac{\partial R_j}{\partial x_i} + [R_i, R_j] = 0 \quad \forall\, i \neq j$$

Exercise: $\exists!$ solution of $(*)$ of the form (under a non-resonance hypothesis akin to the one from Thm. 4 of Lecture 1, page 6.)

$$\psi(\underline{x}) = H(x_1, \dots, x_n) \cdot \prod_{i=1}^{n} x_i^{t_i}$$

unambiguous since t_i's commute

where

$H(\underline{0}) = 1.$, and H is holomorphic near $\underline{x} = \underline{0}$.

Then:

$$\mu_\psi \left(\overset{x_i = 0}{\not{\circlearrowleft}}\ p_0 = (1, 1, \dots, 1) \right) = \exp\left(2\pi\sqrt{-1}\, t_i\right)$$

9. To treat the general case, we need some definitions.

Let D be the (Dynkin) diagram of R. For the purposes of this section type of an edge (i.e. $\frac{\overline{}}{}$ or $\Rightarrow$ or $\equiv$) is immaterial.

Definitions. Two subdiagrams B_1 and B_2 are called <u>orthogonal</u> (notation $B_1 \perp B_2$) if
$$\begin{cases} \text{they do not share any vertex; and} \\ \begin{array}{l} \alpha \in B_1 \\ \beta \in B_2 \end{array} \Rightarrow \alpha \text{ and } \beta \text{ are not connected in } D. \end{cases}$$

We say B_1 and B_2 are <u>compatible</u> if either $B_1 \subset B_2$ or $B_2 \subset B_1$, or $B_1 \perp B_2$.

A <u>nested set</u> on D is a collection $\mathcal{H}$ of pairwise compatible, connected subdiagrams of D.

A <u>maximal nested set on</u> D is a nested set which is maximal w.r.t. inclusion.

10. Example. – $D = A_n$ $\quad \underset{1}{\bullet}\!-\!\underset{2}{\bullet}\!-\!\underset{3}{\bullet} \cdots \underset{n-1}{\bullet}\!-\!\underset{n}{\bullet}$

Connected subdiagrams $= \left\{ \underset{i}{\bullet} - \cdots - \underset{j}{\bullet} \;\middle|\; 1 \leq i \leq j \leq n \right\}$

A maximal nested set for $n=3$

$$\mathcal{F}_1 = \{\, 1 - 2 - 3 \,,\; 1 \,,\; 3 \,\} \;;\quad \mathcal{F}_2 = \{\, 1 - 2 - 3 \,;\; 1 - 2 \,;\; 1 \,\}$$

$$\mathcal{F}_3 = \{\, 1 - 2 - 3 \,;\; 1 - 2 \,;\; 2 \,\}$$

Nested sets vs. bracketings on $x_1 x_2 \cdots x_{n+1}$.

$$i \longleftrightarrow \cdots (x_i x_{i+1}) \cdots$$

$$i - \cdots - j \longleftrightarrow x_1 \cdots x_{i-1} (x_i \cdots x_j \, x_{j+1}) \cdots x_{n+1}.$$

connected subdiagram of A_n

- Compatibility – same for brackets.
- Maximal nested set – complete bracketings

eg. for $n=2$ $((x_1 x_2) x_3)$ or $(x_1 (x_2 x_3))$

$n=3$ $\underset{\updownarrow}{\overset{\mathcal{F}_2 \text{ above}}{}}$ $(((x_1 x_2) x_3) x_4)$, $((x_1 (x_2 x_3)) x_4)$ $\updownarrow \mathcal{F}_3$ above , $((x_1 x_2)(x_3 x_4))$ $\updownarrow \mathcal{F}_1$ above ,

$(x_1 ((x_2 x_3) x_4))$, $(x_1 (x_2 (x_3 x_4)))$.

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

$$\{\, 1 - 2 - 3 \,,\; 2 \,,\; 2 - 3 \,\} \quad \{\, 1 - 2 - 3 \,;\; 2 - 3 \,;\; 3 \,\}$$