0. Set up : We have a (finite) root system $R \subset E^* \setminus \{0\}$.

$\mathfrak{h} := E \otimes_{\mathbb{R}} \mathbb{C}$; $\alpha \in R$ is now viewed as a $\mathbb{C}$-linear form

$\alpha : \mathfrak{h} \to \mathbb{C}$ and $\mathfrak{h}^{reg} := \mathfrak{h} \setminus \bigcup_{\alpha \in R} H_\alpha$ $\quad (H_\alpha = \mathrm{Ker}(\alpha) \subset \mathfrak{h})$.

Let F be another finite dimensional $\mathbb{C}$-vector space and assume that we are given $t_\alpha \in \mathrm{End}(F)$ $\forall \alpha \in R$ s.t. $t_{-\alpha} = t_\alpha$.

$$\nabla := d - \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} t_\alpha = d - \frac{1}{2} \sum_{\alpha \in R} \frac{d\alpha}{\alpha} t_\alpha .$$

Holonomy relations : For every $Y \subset R_+$ maximal such that Span(Y) is two dimensional, we have

$$\left[\sum_{\alpha \in Y} t_\alpha , t_\beta \right] = 0 \quad \forall \beta \in Y.$$

W-equivariance : Assuming we have an action $W \circlearrowright F$

$$w t_\alpha w^{-1} = t_{w(\alpha)} \quad \forall w \in W, \alpha \in R.$$

1. Let D be the Dynkin diagram of R. Recall that we defined the notions of nested and maximal nested sets on D as follows:

- $B_1, B_2 \subset D$ (full) subdiagrams are called <u>orthogonal</u>

  if $\begin{cases} B_1, B_2 \text{ have no vertex in common.} \\ \alpha \in B_1, \beta \in B_2 \Rightarrow \alpha \text{ \& } \beta \text{ are not connected in } D. \end{cases}$

- B_1, B_2 are said to be <u>compatible</u> if either one is contained in the other; or $B_1 \perp B_2$.

- A <u>nested set</u> on D is a set of <u>pairwise compatible</u>, <u>connected</u> subdiagrams of D. A <u>maximal nested set</u> is a nested set which is maximal w.r.t. inclusion.

(§10 of Lecture 10).

e.g. $D = A_n$ $\quad 1 - 2 - 3 \cdots - n{-}1 - n$

Connected subdiagrams $= \underset{i}{\bullet} - \cdots - \underset{j}{\bullet}$ $\quad 1 \leq i \leq j \leq n.$

$\rightsquigarrow$ bracketing on $n+1$ variables $x_1\, x_2 \cdots x_{n+1}$

$\underset{i}{\bullet} - \cdots - \underset{j}{\bullet} \quad \longleftrightarrow \quad x_1 \cdots x_{i-1} (x_i \cdots x_{j+1})\, x_{j+2} \cdots x_{n+1}$

Maximal Nested Sets $\longleftrightarrow$ Complete bracketings

$$\# \, \mathrm{Mns}(A_n) = \frac{1}{n+1}\binom{2n}{n} \quad (n^{\text{th}} \text{ Catalan number}).$$

2. Elementary properties of maximal nested sets on a diagram D:

Let $\mathcal{F}$ = a maximal nested set on D.

Lemma. For $B \in \mathcal{F}$, consider the set

$$\mathcal{F}\big|_B = \{ B' \in \mathcal{F} \; : \; B' \subsetneq B \}.$$

Let $B_1, \ldots, B_k$ be the maximal elements of $\mathcal{F}|_B$. Then

(i) $B_1, \ldots, B_k$ are pairwise orthogonal.

(ii) There is a unique vertex $\alpha \in B$ such that $\alpha \notin B_i \;\; \forall\, i = 1, \ldots, k$.

In fact, $B_1, \ldots, B_k$ then have to be connected components of $B \setminus \{\alpha\}$.

Proof. (i) is obvious ($B_1, \ldots, B_k$ are assumed to be compatible pairwise, and each is maximal w.r.t. inclusion, so $B_i \not\subset B_j$. The only option is $B_i \perp B_j$).

(ii) This is also clear since if $B_1 \cup \dots \cup B_k = B$, then we will contradict their pairwise orthogonality. And if they fail to cover two or more vertices of B, we will contradict the maximality of $\mathcal{F}$. $\square$

Corollary. $|\mathcal{F}| = \#$ vertices of $D = \dim_{\mathbb{C}} \mathfrak{h}^* = \dim_{\mathbb{C}} \mathfrak{h}$.

Remark. The number k from the lemma above can only be 1, 2 or 3 in our case (when D is the Dynkin diagram of a finite root system).

3. Definition: A collection of linear forms $\{x_B\}_{\substack{B \subseteq D \\ \text{connected} \\ \text{subdiagram}}}$; $x_B \in \mathfrak{h}^*$ is called an <u>adapted family</u> if for any m.n.s. $\mathcal{F}$ on D and $B \in \mathcal{F}$; the set of elements $\{x_C\}_{\substack{C \in \mathcal{F} \\ C \subseteq B}}$ form a basis of the subspace $\mathfrak{h}^*_B$ of $\mathfrak{h}^*$ spanned by $\{\alpha_j : j \text{ a vertex of } B\}$.

Remark. - The change of coordinates we are about to describe depend on the choice of such an "adapted family".

The essential point in this definition being that:

$$\left[\begin{array}{l} B \in \mathcal{F} \;;\; B_1, \dots, B_k \text{ as in Lemma 2 above. Then} \\ \dim \mathfrak{h}_B^* = 1 + \sum_{j=1}^{k} \dim \mathfrak{h}_{B_j}^* \quad \text{and we want } x_B \in \mathfrak{h}_B^* \setminus \left(\bigoplus_{j=1}^{k} \mathfrak{h}_{B_j}^* \right). \end{array}\right.$$

e.g. we can take $x_B = \sum_{j \in B} \alpha_j \quad \forall\, B \subseteq D$ connected subdiagram.

Or, let $R_B = \mathfrak{h}_B^* \cap R$ (= root system gen. by B) and

$$x_B = \sum_{\alpha \in R_B \cap R_+} \alpha \quad \text{(sum of +ve roots of the sub-root system generated by } B\text{).}$$

4. Let $\mathcal{F}$ be a maximal nested set. Let $\mathcal{U} = \mathbb{C}^{\mathcal{F}}$ be $|\mathcal{F}|$-dimensional affine space with coordinates $\{u_B : B \in \mathcal{F}\}$.

$(|\mathcal{F}| = \dim \mathfrak{h}^*)$

$$\begin{array}{rccl} \rho_{\mathcal{F}} : & \mathcal{U} & \longrightarrow & \mathfrak{h} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & \underline{u} & \longmapsto & \rho_{\mathcal{F}}(\underline{u}) \text{ uniquely defined by:} \end{array}$$

$$x_B\left(\rho_{\mathcal{F}}(\underline{u})\right) = \prod_{\substack{C \in \mathcal{F} \\ B \subseteq C}} u_C \qquad \forall\, B \in \mathcal{F}.$$

Note: $\{x_B\}_{B \in \mathcal{F}}$ is a basis of $\mathfrak{h}^*$ and the equation given above singles out $\rho_{\mathcal{F}}(\underline{u}) \in \mathfrak{h}$ by prescribing the values of $x_B(\rho_{\mathcal{F}}(\underline{u}))$ $(B \in \mathcal{F})$ in terms of the u-coordinates of $\mathbb{C}^{\mathcal{F}}$.

The relation $\displaystyle x_B = \prod_{\substack{G \in \mathcal{F} \\ B \subseteq G}} u_G$ can be easily inverted to give

$$u_B = \begin{cases} x_D & \text{if } B = D \\ \dfrac{x_B}{x_{C(B)}} & \text{o/w} \end{cases}$$

; where $C(B) \in \mathcal{F}$ is the unique minimal element properly containing B.

Example 1. (B_2) $\quad R_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$

Adapted family:	$x_1 = \alpha_1$	$x_2 = \alpha_2$	$x_{1-2} = \alpha_1+\alpha_2$
Connected Subdiagram:	$\bullet_1$	$\bullet_2$	$\bullet_1 - \bullet_2$

Take $\mathcal{F} = \{\bullet_1 \,;\, \bullet_1 - \bullet_2\}$ $\rightsquigarrow$ $\rho : \mathbb{C}^2 \longrightarrow \mathfrak{h}$ by

$$\alpha_1 = u_1 \cdot u_2$$
$$\alpha_1 + \alpha_2 = u_2$$

Inverse
$$u_2 = \alpha_1 + \alpha_2$$
$$u_1 = \frac{\alpha_1}{\alpha_1+\alpha_2}$$

Example 2. (A_3) $\quad \bullet_1 - \bullet_2 - \bullet_3$

$$R_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}$$

Adapted family

x_B	α_1	α_2	α_3	$\alpha_1+\alpha_2$	$\alpha_2+\alpha_3$	$\alpha_1+\alpha_2+\alpha_3$
$B \subseteq D$ conn.	$\underset{1}{\bullet}$	$\underset{2}{\bullet}$	$\underset{3}{\bullet}$	$\underset{1}{\bullet}\!-\!\!-\!\underset{2}{\bullet}$	$\underset{2}{\bullet}\!-\!\!-\!\underset{3}{\bullet}$	$\underset{1}{\bullet}\!-\!\!-\!\underset{2}{\bullet}\!-\!\!-\!\underset{3}{\bullet}$

Take $\mathcal{F} = \left\{ \underset{1}{\bullet} \; ; \; \underset{1}{\bullet}\!-\!\!-\!\underset{2}{\bullet} \; ; \; \underset{1}{\bullet}\!-\!\!-\!\underset{2}{\bullet}\!-\!\!-\!\underset{3}{\bullet} \right\} \rightsquigarrow \rho : \mathbb{C}^3 \longrightarrow \mathfrak{h}$ by

$$\begin{aligned} \alpha_1 &= u_1 u_2 u_3 \\ \alpha_1 + \alpha_2 &= u_2 u_3 \\ \alpha_1+\alpha_2+\alpha_3 &= u_3 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} u_1 &= \frac{\alpha_1}{\alpha_1+\alpha_2} \\ u_2 &= \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\alpha_3} \\ u_3 &= \alpha_1+\alpha_2+\alpha_3 \end{aligned}$$

5. Pull-back of an arbitrary root. Let $\alpha \in R$. Let $B \in \mathcal{F}$ be the unique minimal element s.t. $\alpha \in R_B$. Since $\{x_{B'}\}_{\substack{B' \subseteq B \\ B' \in \mathcal{F}}}$ form a basis of $\mathfrak{h}_B^*$, we can write (with $a_B \neq 0$)

$$\alpha = \sum_{\substack{B' \in \mathcal{F} \\ B' \subseteq B}} a_{B'} \, x_{B'} \qquad (a_{B'} \in \mathbb{C} \; ; \; a_B \neq 0)$$

$$= a_B \, x_B \cdot \left(1 + \sum_{\substack{B' \in \mathcal{F} \\ B' \subsetneq B}} \frac{a_{B'}}{a_B} \, \frac{x_{B'}}{x_B} \right)$$

Note: $B' \subsetneq B \Rightarrow \dfrac{x_{B'}}{x_B} = \dfrac{\prod_{B' \subseteq G} u_G}{\prod_{B \subseteq G} u_G} = \prod_{B' \subseteq G \subsetneq B} u_G$ is a monomial in u-variables

Hence $$\alpha = a_B\, x_B \cdot \left(1 + \sum_{\substack{B' \in \mathcal{F} \\ B' \subsetneq B}} \frac{a_{B'}}{a_B} \prod_{\substack{G \in \mathcal{F} \\ B' \subseteq G \subsetneq B}} u_G \right)$$

$$x_B = \prod_{\substack{G \in \mathcal{F} \\ B \subseteq G}} u_G$$

<u>Proposition</u> $\forall\ \alpha \in R$; $\exists$ a polynomial $P_\alpha(\underline{u})$ s.t.

$$\alpha = a_B \cdot \prod_{\substack{G \in \mathcal{F} \\ B \subseteq G}} u_G \cdot P_\alpha(\underline{u}) \qquad \left(\begin{array}{c} \text{here } B \in \mathcal{F} \text{ is} \\ \text{min'l s.t.} \\ \alpha \in R_B \end{array} \right)$$

- $P_\alpha(\underline{u})$ only depends on variables $\{u_{B'}\}_{\substack{B' \in \mathcal{F} \\ B' \subsetneq B}}$
- $P_\alpha(0) = 1$.

e.g. B_2 example from page 6 above :

$$2\alpha_1 + \alpha_2 = u_2 \, \overbrace{(1 + u_1)}^{P_{2\alpha_1+\alpha_2}}$$

A_3 example from the same section

$$\alpha_3 = u_3 \, \overbrace{(1 - u_2)}^{P_{\alpha_3}}$$

$$\alpha_2 = u_2 u_3 \, \overbrace{(1 - u_1)}^{P_{\alpha_2}}$$

$$\alpha_2 + \alpha_3 = u_3 \left(1 - u_2 + u_2 - u_1 u_2 \right) = u_3 \, \overbrace{(1 - u_1 u_2)}^{P_{\alpha_2+\alpha_3}}.$$