Lecture 12

0. Recall the set up from Lecture 11, §0, page 1.

D = (Dynkin) diagram of the root system $R \subset E^* \setminus \{0\}$.

$\left(\mathfrak{h} = E \underset{\mathbb{R}}{\otimes} \mathbb{C} \; ; \quad H_\alpha = \mathrm{Ker}(\alpha) \subset \mathfrak{h} \quad \forall \alpha \in R \right).$

For every connected subdiagram $B \subseteq D$; $\quad x_B = \sum_{i \in B} \alpha_i \quad$ (choice of adapted family)

$\mathrm{Mns}(D)$ = maximal nested sets on D.

(1) For any $\mathcal{S} \in \mathrm{Mns}(D)$ and $B \in \mathcal{S}$; $\{x_{B'} : B' \in \mathcal{S}, B' \subseteq B\}$ is a basis of $\mathfrak{h}_B^*$. (Recall: $\mathfrak{h}_B^* = \mathrm{Span}\{\alpha_i : i \in B\} \subseteq \mathfrak{h}^*$).

(2) For any $\mathcal{S} \in \mathrm{Mns}(D)$, we defined

$$\rho_{\mathcal{S}} : \mathbb{C}^{\mathcal{S}} \longrightarrow \mathfrak{h}$$
$$(u_B : B \in \mathcal{S}) \longmapsto \left\{ x_B = \prod_{\substack{C \in \mathcal{S} \\ B \subseteq C}} u_C \quad ; \; \forall B \in \mathcal{S} \right\}$$

and proved that, for every $\alpha \in R_+$, there exists a polynomial $P_\alpha(\underline{u})$ in variables $\{u_{B'} : B' \in \mathcal{S}, B' \subsetneq B_\alpha\}$; where

$B_\alpha \in \mathcal{S}$ is minimal so that $\alpha \in \mathfrak{h}_{B_\alpha}^*$.

- $P_\alpha(0) = 1$
- $\alpha \sim \prod_{\substack{C \in \mathcal{S} \\ B_\alpha \subseteq C}} u_C \, . \, P_\alpha(\underline{u})$ (i.e. they are non-zero scalar multiple of each other).

1. Corollary of the computation above:

(a) $\underset{\displaystyle H_\alpha}{\overset{\parallel}{\mathrm{Ker}(\alpha)}} \subset \mathfrak{h} \rightsquigarrow \rho_{\mathcal{S}}^{-1}(H_\alpha) = \{u_C = 0\}_{\substack{C \in \mathcal{S} \\ B_\alpha \subseteq C}} \cup \{P_\alpha = 0\}$

(b) $$\frac{d\alpha}{\alpha} = \sum_{\substack{C \in \mathcal{S} \\ B_\alpha \subseteq C}} \frac{du_C}{u_C} + \frac{dP_\alpha}{P_\alpha}$$

2. Returning back to our connection $\nabla = d - \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} t_\alpha$, and substituting (b); we get

$$\nabla = d - \sum_{B \in \mathcal{S}}' \frac{du_B}{u_B} \cdot \Big(\sum_{\substack{\alpha \in R_+ \\ \text{s.t. } B_\alpha \subseteq B}} t_\alpha \Big) - \underbrace{\sum_{\alpha \in R_+} \frac{dP_\alpha}{P_\alpha} t_\alpha}_{\substack{\text{regular near} \\ \underline{u} = \underline{0}.}}$$

Set $t_B = \sum_{\substack{\alpha \in R_+ \\ B_\alpha \subseteq B}} t_\alpha = \sum_{\alpha \in \mathfrak{h}_B^* \cap R_+} t_\alpha$, so that

$$\nabla = d - \sum_{B \in \mathcal{S}} \frac{du_B}{u_B} \cdot t_B - \sum_{\alpha \in R_+} \frac{dP_\alpha}{P_\alpha} t_\alpha$$

Easy exercise: holonomy relations for $\{t_\alpha\}$ imply that $[t_{B_1}, t_{B_2}] = 0 \quad \forall\, B_1, B_2 \in \mathcal{S}$.

Hence, we return to the "normal crossing" example from Lecture 10, § 8, page 5. That is, we have a unique solution of $\nabla\psi = 0$ of the form

$$\psi_{\mathcal{S}} = H_{\mathcal{S}}(\underline{u}) \cdot \prod_{B \in \mathcal{S}} u_B^{t_B} ,$$

where $H_{\mathcal{S}}(\underline{u})$ is holomorphic near $\underline{u} = \underline{0}$ and $H_{\mathcal{S}}(\underline{0}) = 1$.

Remark. Using $x_B = \prod_{\substack{G \in \mathcal{S} \\ B \subseteq G}} u_G \quad \longleftrightarrow \quad u_B = \begin{cases} x_B & \text{if } B = D \\ \dfrac{x_B}{x_{c(B)}} & \text{o/w} \end{cases}$

($c(B) \in \mathcal{S}$ is the unique minimal element properly containing B).

we get: $\displaystyle\prod_{B \in \mathcal{S}} u_B^{t_B} = \prod_{B \in \mathcal{S}} x_B^{r_B}$ where $r_B = t_B - \sum_{i=1}^{k} t_{B_i}$;

and $\{B_1, \dots, B_k\}$ = maximal elements of $\mathcal{S}|_B = \{B' \in \mathcal{S} \mid B' \subsetneq B\}$

(see Lemma 2 of Lecture 11, page 3).

3. De Concini - Procesi associators.

Note that, by our choice of $\{x_B\}_{\substack{B \subseteq D \\ \text{connected}}}$, these linear forms are real and positive on $\mathcal{C}^{\circ} = \{h \in \mathfrak{h} : \alpha_i(h) \in \mathbb{R}_{>0}\}$.

Assuming the standard branch of log is taken in the definition of the multivalued function $\prod_{B \in \mathcal{S}} u_B^{t_B} = \prod_{B \in \mathcal{S}} x_B^{r_B}$ above, we obtain, for any maximal nested set $\mathcal{S}$, a solution $\psi_{\mathcal{S}}$ of $\nabla \psi = 0$, valid on $\mathcal{C}^\circ$.

<u>DCP associator</u> is defined as $\Phi_{\mathcal{G}\mathcal{F}} = \psi_{\mathcal{G}}(y)^{-1} \psi_{\mathcal{F}}(y)$ (for any $y \in \mathcal{C}^\circ$) for all $\mathcal{G}, \mathcal{F} \in \mathrm{Mns}(\mathcal{D})$.

Remarks and references. The change of coordinates described here has been used by: Drinfeld (Quasi-Hopf algebras - Leningrad 1990) in the context of KZ equations - which involve type A hyperplane arrangement.

- Cherednik (Monodromy repns for generalised KZ eq^{n}s and Hecke algebras - Publ. RIMS 1990) - arbitrary root system.

We have followed De Concini - Procesi

- Wonderful models of subspace arrangements (Selecta Math. 1995)
- Hyperplane arrangements and holonomy eq^{n}s (" " ")

DCP theory is adapted to more general situations of <u>any</u> hyperplane arrangement not necessarily coming from a root system.

4. Now we will state the geometric side of this story, as appeared in [DCP - Wonderful].

- Let $\mathrm{Irr}(R)$ = Set of irreducible root-sub-systems of R.

(Remark - up to W-action - an irreducible root-sub-system of R is of the form $R_B = \mathfrak{h}_B^* \cap R$, where $B \subseteq D$ is a connected sub diagram).

- For any $A \in \mathrm{Irr}(R)$, let $A^{\perp} = \left\{ h \in \mathfrak{h} \;\middle|\; \begin{array}{l} \alpha(h) = 0 \\ \forall \alpha \in A \end{array} \right\}$.

Then we have a canonical map

$$\mathfrak{h} \setminus A^{\perp} \longrightarrow \mathbb{P}_A = \mathbb{P}(\mathfrak{h}/A^{\perp})$$

(projective space of lines in $\mathfrak{h}/A^{\perp}$)

Note that $\mathfrak{h}^{reg} \subset \mathfrak{h} \setminus A^{\perp}$ for any $A \subset R$.

- Thus we obtain $\eta : \mathfrak{h}^{reg} \longrightarrow \mathfrak{h} \times \prod_{A \in \mathrm{Irr}(R)} \mathbb{P}_A$.

Define $Y_R :=$ Closure of $\eta(\mathfrak{h}^{reg}) \subset \mathfrak{h} \times \prod_{A \in \mathrm{Irr}(R)} \mathbb{P}_A$.

Set $\pi : Y_R \to \mathfrak{h}$ to be the composition

$$Y_R \subset \mathfrak{h} \times \prod_{A \in \mathrm{Irr}(R)} \mathbb{P}_A \xrightarrow[pr_1]{\text{projection}} \mathfrak{h}.$$

Theorem. – (i) Y_R is smooth irreducible algebraic variety.

(ii) Let $\mathcal{D} = \bigcup_{\alpha \in R} H_\alpha \subset \mathfrak{h}$ and $\tilde{\mathcal{D}} = \pi^{-1}(\mathcal{D}) \subset Y_R$. Then

π induces an isomorphism $Y_R \setminus \tilde{\mathcal{D}} \longrightarrow \mathfrak{h} \setminus \mathcal{D} = \mathfrak{h}^{reg}$.

(iii) $\tilde{\mathcal{D}}$ is a normal crossing divisor. Its irreducible components are : $\{ \mathcal{D}_B = \pi^{-1}(B^\perp) \mid B \in \mathrm{Irr}(R) \}$.

(iv) A subset of components of $\tilde{\mathcal{D}}$, say $\{ \mathcal{D}_B \mid B \in \mathcal{T} \subset \mathrm{Irr}(R) \}$, intersect non-trivially $\iff$ $\mathcal{T}$ is a nested set.

5. The choices of adapted family and a maximal nested set are needed to describe open charts of Y_R.

Given a maximal nested set $\mathcal{S}$, we have a monomial map

$$\rho_{\mathcal{S}} : \mathbb{C}^{\mathcal{S}} \longrightarrow \mathfrak{h} \qquad \Big(x_B = \prod_{\substack{C \in \mathcal{S} \\ B \subseteq C}} u_C \quad \forall\, B \in \mathcal{S} \Big)$$

Let $P_\alpha(\underline{u})$ be the polynomials introduced in Prop. 5, page 8 of Lecture 11.

$$\mathcal{U}_{\mathcal{S}} := \mathbb{C}^{\mathcal{S}} \setminus \bigcup_{\alpha \in R} \{ P_\alpha = 0 \}$$

Theorem. (i) $\{ \mathcal{U}_S \}_{S\,:\,\text{max'l nested set}}$ is an open cover of Y_R.

(ii) Let $B \in \mathrm{Irr}(R)$. Then $\mathcal{D}_B \cap \mathcal{U}_{\mathcal{S}} \neq \phi$ if and only if $B \in \mathcal{S}$ in which case $\mathcal{D}_B \cap \mathcal{U}_{\mathcal{S}} = \{u_B = 0\}$ (coordinate hyperplane)

In particular, maximal intersections of divisors $\{\mathcal{D}_B\}_{B \in \mathrm{Irr}(R)}$ are given by

$$\bigcap_{B \in \mathcal{S}} \mathcal{D}_B \cap \mathcal{U}_{\mathcal{S}} = \{(\underline{0})\}$$

point in $\mathcal{U}_{\mathcal{S}}$ with all coordinates $u_A = 0 \quad (A \in \mathcal{S})$.

6. How does $\mathcal{U}_{\mathcal{S}} \overset{i}{\subset} Y_R$? Since Y_R admits projections

$$\begin{array}{ccc} Y_R & \xrightarrow{\pi_A} & \mathbb{P}_A \\ {\scriptstyle \pi}\downarrow & & \\ \mathfrak{h} & & \end{array}$$

$\forall\, A \in \mathrm{Irr}(R)$; it is enough to see how the compositions $\pi \circ i$, $\pi_A \circ i$ work.

- $\pi \circ i = \rho_{\mathcal{S}}\big|_{\mathcal{U}_{\mathcal{S}}} : \mathcal{U}_{\mathcal{S}} (\subset \mathbb{C}^{\mathcal{S}}) \to \mathfrak{h}$.

- If $A \in \mathcal{S}$, we can take $\{x_B : B \in \mathcal{S},\ B \subseteq A\}$ as homogeneous coordinates on $\mathbb{P}_A$; in which

$$\pi_A \circ i(\underline{u}) = \Big[\ \underbrace{\prod_{\substack{B \subseteq G \\ G \subsetneq A}} u_G}_{x_B^{th}\ \text{term}} : \underbrace{1}_{x_A^{th}\ \text{term}}\ \Big]$$

- In general we need to first prove a small lemma.

Defn. For any subset $A \subset \mathfrak{h}^*$, let $\rho_S(A) =$ smallest $B \in \mathcal{S}$ s.t. $A \subseteq B$.

Lemma. For any $A \in \mathrm{Irr}(R)$, there exists $\alpha \in A$ s.t.

$$\rho_S(\alpha) = \rho_S(A).$$

[Proof is left as an exercise.] Note that for any $\beta \in A$, $\rho_S(\beta) \subset \rho_S(A)$.

Now we can describe $\pi_A \circ i(\underline{u}) \in \mathbb{P}_A$ in terms of homogeneous coordinates corresponding to a basis $\{\alpha^{(j)}\}_{1 \le j \le \dim \mathfrak{h}_A^*} \subset A$ of $\mathfrak{h}_A^*$; where $\alpha^{(1)} = \alpha$.

$$\pi_A \circ i(\underline{u}) = \Big[\underbrace{P_\alpha(u)}_{\substack{\alpha^{(1)}\text{-st coord} \\ \neq 0 \\ \forall\, \underline{u} \in \mathcal{U}_S}} ; \prod_{\substack{G \in S \\ \rho_S(\alpha^{(j)}) \subseteq G \subsetneq \rho_S(A)}} u_G \cdot P_{\alpha^{(j)}}(\underline{u}) \Big]$$