Lecture 13

Lie algebras and their representations

0. Basic definitions. A <u>Lie algebra</u> (over $\mathbb{C}$) is a vector space $\mathfrak{g}$, over $\mathbb{C}$, together with a bilinear map (called commutator, or just <u>Lie bracket</u>) $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying

- (skew symmetry) $[x,y] = -[y,x] \quad \forall\, x,y \in \mathfrak{g}$.
- (Jacobi identity) $\forall\, x,y,z \in \mathfrak{g}$ we have

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0.$$

Let $\mathfrak{g}$ be a Lie algebra. A <u>representation of $\mathfrak{g}$</u> is a vector space V over $\mathbb{C}$, together with a linear map $\mathfrak{g} \xrightarrow{\pi} \mathrm{End}_{\mathbb{C}}(V)$ s.t.

$$\pi([x,y]) = \pi(x)\pi(y) - \pi(y)\pi(x) \quad \forall\, x,y \in \mathfrak{g}.$$

A <u>homomorphism of Lie algebras</u> $f : \mathfrak{g} \to \mathfrak{g}'$ (where $\mathfrak{g}$ and $\mathfrak{g}'$ are Lie algebras) is a linear map satisfying $f([x,y]_{\mathfrak{g}}) = [f(x), f(y)]_{\mathfrak{g}'}$.

$(\forall\, x,y \in \mathfrak{g})$.

1. Remarks. - (i) Given any vector space V over $\mathbb{C}$, the space of all linear endomorphisms $V \to V$, denoted by $\mathrm{End}_{\mathbb{C}}(V)$ above, has a natural structure of a Lie algebra:

$$A, B \in \mathrm{End}(V) \rightsquigarrow [A,B] := AB - BA.$$

We will denote this Lie algebra by $\mathfrak{gl}(V)$; or $\mathfrak{gl}_n(\mathbb{C})$ if V is n-dimensional and we have chosen a basis of V.

(ii) A representation of $\mathfrak{g}$ (on a vector space V) is nothing but a homomorphism of Lie algebras $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

(iii) For any $x \in \mathfrak{g}$, consider the linear map:

$$\begin{aligned} \mathrm{ad}(x) : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ y &\longmapsto [x,y] \end{aligned} \qquad \text{(called \underline{adjoint})}$$

which gives rise to $\begin{aligned} \mathrm{ad} : \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ x &\longmapsto \mathrm{ad}(x) \end{aligned}$. The <u>Jacobi identity</u> is then equivalent to requiring that ad is a homomorphism of Lie algebras and hence defines a <u>representation of $\mathfrak{g}$ on itself</u>.

2. Operations on representations.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. Given V_1, V_2 two representations of $\mathfrak{g}$

$$\pi_j : \mathfrak{g} \longrightarrow \mathfrak{gl}(V_j) \; ; \; j = 1, 2 \, ,$$

we define:

- <u>Direct Sum</u> $\quad V_1 \oplus V_2$ = direct sum of vector spaces (as a vector space)

$$\begin{aligned} \pi_{V_1 \oplus V_2} : \mathfrak{g} &\longrightarrow \mathfrak{gl}(V_1 \oplus V_2) \\ x &\longmapsto \left[\begin{array}{c|c} \pi_1(x) & 0 \\ \hline 0 & \pi_2(x) \end{array} \right] \end{aligned}$$

- Tensor product: $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V_1 \otimes_{\mathbb{C}} V_2)$ is defined by:

$$\pi(x)\cdot(v_1 \otimes v_2) = \pi_1(x)(v_1) \otimes v_2 + v_1 \otimes \pi_2(x)(v_2)$$

$\forall\ x \in \mathfrak{g}$, $v_1 \in V_1$ and $v_2 \in V_2$

[This is often written as $\pi(x) = \pi_1(x) \otimes Id_{V_2} + Id_{V_1} \otimes \pi_2(x)$.]

For a representation V of $\mathfrak{g}$, $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, its dual representation ($\mathfrak{g} \circlearrowright V^*$) is defined by $\mathfrak{g} \xrightarrow{\pi^*} \mathfrak{gl}(V^*)$:

$$\left(\pi^*(x)(\xi)\right)(v) = -\xi\left(\pi(x)(v)\right) \quad \forall\ x \in \mathfrak{g};\ \xi \in V^*;\ v \in V.$$

3. Let $\mathfrak{g}$ be a Lie algebra and V a representation of $\mathfrak{g}$. We say V is irreducible if 0 and V are the only invariant subspaces under $\mathfrak{g}$ action. That is, $\left.\begin{cases} V' \subseteq V \\ \pi(x)(V') \subseteq V',\ \forall x \in \mathfrak{g} \end{cases}\right\} \Rightarrow V' = 0 \text{ or } V.$

[0 is not considered irreducible.]

V is said to be indecomposable if $V \cong V_1 \oplus V_2$ as $\mathfrak{g}$-repns (for $V_1, V_2 \subset V$ subrepns.) implies that either $V_1 = 0$ or $V_2 = 0$.

[We will mostly work with finite-dimensional representations.]

Given two representations V_1, V_2 of $\mathfrak{g}$; a linear map $f : V_1 \to V_2$ is said to be a homomorphism of $\mathfrak{g}$-representations (or, $\mathfrak{g}$-intertwiner,

or just $\mathfrak{g}$-linear) if $f(\pi_1(x)(v)) = \pi_2(x)(f(v))$
$\forall\, x \in \mathfrak{g}$ and $v \in V$.

Remarks (i) If $\mathfrak{g} \circlearrowright V_1, V_2$, then we can define $\mathfrak{g} \circlearrowright \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$

by $$(x \cdot A)(v_1) = \pi_2(x)\big(A(v_1)\big) - A\big(\pi_1(x)(v_1)\big)$$
$\forall\, x \in \mathfrak{g},\ A \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2),\ v_1 \in V_1$.

Thus $$\mathrm{Hom}_{\mathfrak{g}}(V_1, V_2) = \{ A \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2) \mid x \cdot A = 0\ \forall x \in \mathfrak{g} \}$$

(ii) $\mathfrak{g} \circlearrowright V$. We define $V^{\mathfrak{g}}$ ($\mathfrak{g}$-invariants) to be
$$V^{\mathfrak{g}} = \{ v \in V \mid x \cdot v = 0\ \forall x \in \mathfrak{g} \}.$$

4. Schur's Lemma. Let V_1 and V_2 be two irreducible representations of $\mathfrak{g}$ and let $f : V_1 \to V_2$ be a morphism of $\mathfrak{g}$-representations.
Then: either $f \equiv 0$, or f is an isomorphism.

[Proof: $\mathrm{Ker}(f) = \{ v_1 \in V_1 : f(v_1) = 0 \} \subseteq V_1$ is a subrepresentation

$\underset{(V_1 \text{ irred.})}{\Rightarrow}$ $\mathrm{Ker} f = V_1$ (i.e. $f \equiv 0$) or $\mathrm{Ker} f = 0$. If $\mathrm{Ker} f = 0$, we get

$f(V_1) \subseteq V_2$ subrepn. Since V_2 is irreducible & $V_1 \neq 0$, $f(V_1) = V_2$.
Hence f is both injective & surjective,]

Schur's Lemma continued. Assume V is a finite-dimensional irreducible representation of $\mathfrak{g}$; and let $f: V \to V$ be a $\mathfrak{g}$-intertwiner. Then $\exists\, \lambda \in \mathbb{C}$ s.t. $f = \lambda \cdot \mathrm{Id}_V$.

Proof. As $f \in \mathrm{End}_{\mathbb{C}}(V)$ and V is finite-dimensional (and $\mathbb{C}$ is algebraically closed); we can find an eigenvector of f: i.e., $\exists \underset{v \neq 0}{v \in V}$ and $\lambda \in \mathbb{C}$ s.t. $f(v) = \lambda v$.

Now $\mathrm{Ker}(f - \lambda \cdot \mathrm{Id}_V) \subset V$ is a subrepn. since f is a $\mathfrak{g}$-intertwiner and $0 \neq v \in \mathrm{Ker}(f - \lambda \cdot \mathrm{Id}_V)$ - so it cannot be zero. By irreducibility of V; $\mathrm{Ker}(f - \lambda \cdot \mathrm{Id}_V) = V$. $\square$

5. <u>Lie algebra $\mathfrak{sl}_2(\mathbb{C})$</u> is the Lie algebra of 2×2 matrices with trace zero. Thus, it is 3-dimensional with basis

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Commutators: $[h, e] = 2e$ $\qquad [e, f] = h$

$[h, f] = -2f$

Therefore, a representation of sl_2 is nothing but a vector space V together with 3 linear maps $\pi(h), \pi(e), \pi(f) \in \mathrm{End}_{\mathbb{C}}(V)$ satisfying $[\pi(h), \pi(e)] = 2\pi(e)$

$[\pi(h), \pi(f)] = -2\pi(f)$ and $[\pi(e), \pi(f)] = \pi(h)$.

Proposition. Let V be a finite-dimensional, irreducible representation of sl_2. Let $\dim V = n+1$ where $n \in \mathbb{Z}_{\geq 0}$. Then there exists a basis of V $\{v_0, v_1, \dots, v_n\}$ s.t.

$$h.v_i = (n-2i)v_i \quad ; \quad f\,v_i = (i+1)\,v_{i+1} \quad ; \quad e.v_i = (n-i+1)\,v_{i-1}$$

$\forall\ 0 \leq i \leq n$; with the convention that $v_{-1} = v_{n+1} = 0$.

Proof. Since $h \in \mathrm{End}_{\mathbb{C}}(V)$, and V is f.d., we can find an eigenvector $v \neq 0$; $v \in V$ of h with eigenvalue $\lambda \in \mathbb{C}$. That is, $h.v = \lambda.v$.

Note : $h.e = e.(h+2)$ $\Rightarrow$ $e^k.v$ is an eigenvector of h with eigenvalue $\lambda + 2k$.

$(h.f = f.(h-2))$

[Remark.- vectors with distinct eigenvalues $\rightsquigarrow$ linearly independent.]

Since V is finite-dim'l, we can find k $(k \geq 0)$ s.t. $e^k.v \neq 0$ and $e^{k+1}.v = 0$. Set $v_0 = e^k.v$ and let $\mu \in \mathbb{C}$ be its h-eigenvalue

$(\mu = \lambda + 2k)$

For each $l \geq 0$; define $v_l = \frac{f^l}{l!} v_0$. Then we have

$$h.v_l = (\mu - 2l)\, v_l \qquad \text{and} \qquad f.v_l = (l+1)\, v_{l+1}.$$

We will prove (by induction on l) that $e.v_l = (\mu - l + 1)\, v_{l-1}$

(Note $e.v_0 = 0$)

$l = 1.$ $\quad e\, v_1 = e.(f.v_0) = f.\underbrace{(e.v_0)}_{0} + h.v_0 = \mu.v_0.$

$l > 1.$ $\quad e.v_l = \frac{1}{l!}\left(e f^l\right) v_l = \frac{1}{l!}\left(fe + h\right) f^{l-1}.v$

$$= \frac{1}{l} f.(e.v_{l-1}) + \frac{1}{l}\left(\mu - 2(l-1)\right) v_{l-1}$$

$$= \frac{1}{l} f.\left((\mu - l + 2)\, v_{l-2}\right) + \frac{1}{l}\left(\mu - 2l + 2\right) v_{l-1}$$

$$= \frac{1}{l}\Big((l-1)(\mu - l + 2) + \mu - 2l + 2 \Big) v_{l-1}$$

$$= \frac{1}{l}\Big(l.\mu - l(l-1) \Big) v_{l-1} = (\mu - l + 1)\, v_{l-1}.$$

Let n be smallest ($n \in \mathbb{Z}_{\geq 0}$) s.t. $v_n \neq 0$ and $v_{n+1} = 0$.
(exists by finite-dimensionality of V).

Then (i) $0 = e.v_{n+1} = (\mu - n)\, v_n \Rightarrow \mu = n \in \mathbb{Z}_{\geq 0}.$

(ii) $\mathrm{Span}\{v_0, \ldots, v_n\} \subseteq V$ is a subrepn. hence $= V$ by irreducibility of V.