Lecture 14

0. Recall: $\mathfrak{sl}_2(\mathbb{C})$ = Lie algebra of 2×2 traceless matrices

$$= \mathrm{Span}\{h, e, f\} \text{ with } \left\{\begin{array}{l}[h,e] = 2e \\ [h,f] = -2f \\ [e,f] = h\end{array}\right\}$$

We proved that if V is an irreducible finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$, then $V \cong L_n$ ($n \in \mathbb{Z}_{\geq 0}$ is given by $n = \dim V - 1$).

L_n : basis $\{v_0, \ldots, v_n\}$ and $\mathfrak{sl}_2$-action is given by

$$h v_j = (n-2j)v_j \; ; \quad f.v_j = (j+1)v_{j+1} \; ; \quad e.v_j = (n-j+1)v_{j-1}.$$

We record here an important corollary of our proof (Lecture 13, page 7)

1. Corollary. – Let V be a finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$.
Let $v \in V$ be a non-zero vector s.t. $h.v = \mu . v$ and $e.v = 0$.
$(\mu \in \mathbb{C})$

Then $\mu \in \mathbb{Z}_{\geq 0}$ and $f^{\mu+1}.v = 0$.

[Proof. We have the following identity in $\mathrm{End}(V)$:

$$e.f^k = f^k.e + k f^{k-1}(h-k+1).$$

Thus, if $m \in \mathbb{Z}_{\geq 0}$ is (smallest) s.t. $f^m v \neq 0$ and $f^{m+1} v = 0$, then

$$0 = e.f^{m+1}.v = \left(f^{m+1}e + (m+1)f^m(h-m)\right)v$$

$$\underset{(e.v=0)}{=} (m+1)(\mu-m)(f^m v) \quad \Rightarrow \mu = m. \qquad \square\,]$$

2. Recall : $\mathfrak{g} \circlearrowright V \rightsquigarrow \mathfrak{g} \circlearrowright V^*$ by $(x \cdot \xi)(v) = -\xi(x \cdot v)$.
$\forall\, x \in \mathfrak{g},\ \xi \in V^*,\ v \in V$

The following lemma is very easy to prove - (left as an exercise)

Lemma. - $\forall\, n \in \mathbb{Z}_{\geq 0}$, $L_n^* \cong L_n$

3. Casimir element. For $sl_2 \circlearrowright V$ a (finite-dimensional) repn.
$\pi : sl_2 \longrightarrow gl(V) = \mathrm{End}(V)$, define $C_V \in \mathrm{End}(V)$ by

$$C_V = \frac{h^2}{2} + ef + fe \quad \Big(\text{more precisely } \frac{\pi(h)^2}{2} + \pi(e)\pi(f) + \pi(f)\pi(e)\Big)$$

Lemma. $\forall\, x \in sl_2$, $[x, C_V] = 0$.

Proof. (The following computation takes place in $\mathrm{End}(V)$. I am omitting π from these expressions for simplicity).

$$[h, C] = \underbrace{\cancel{[h, \tfrac{h^2}{2}]}}_{0} + [h,e]f + e[h,f] + [h,f]e + f[h,e]$$
$$= +2ef - 2ef - 2fe + 2fe = 0.$$

$$[e, C] = \frac{1}{2}[e, h^2] + e[e,f] + [e,f]e$$
$$= \frac{1}{2}\Big(h[e,h] + [e,h]h\Big) + eh + he$$
$$= -(he + eh) + eh + he = 0.$$

$\square$

Similarly for f.

4. Since C is a central element, it must act as a scalar on any f.d. irreducible representation (by Schur's Lemma).

- $C\big|_{L_n} = \dfrac{n(n+2)}{2} \cdot Id_{L_n}$ for every $n \in \mathbb{Z}_{\geq 0}$.

[$C = \frac{h^2}{2} + ef + fe = \frac{h^2}{2} + h + 2fe$ (using $ef = fe + h$)

$\Rightarrow C \cdot v_0 = \left(\frac{h^2}{2} + h\right) v_0 = \left(\frac{n^2}{2} + n\right) v_0$.]

(as $e.v_0 = 0$)

5. Theorem. Every finite-dimensional sl_2-representation is a direct sum of irreducible representations.

Equivalently: every short exact sequence $0 \to V' \to V \to V'' \to 0$ of sl_2-repns. (finite-dim'l) splits (i.e. $V \cong V' \oplus V''$ as sl_2-repn.)

[The fact that the two statements are equivalent is an easy induction on dimension argument, which I am leaving out.]

Proof. (Sketch): It is enough to consider the case of irreducible f.d. repns. That is, we will prove that every short exact seq.

$$0 \longrightarrow L_n \xrightarrow{i} V \xrightarrow{\pi} L_m \to 0 \qquad m, n \in \mathbb{Z}_{\geq 0}.$$

splits.

- Case $m \neq n$. By taking duals, if necessary, we assume that $n < m$.

In this case $\overline{V} := \{v \in V \mid C\cdot v = \frac{m(m+2)}{2} v\} \subseteq V$ is a subrepresentation which maps, under π, isomorphically to L_m.

$\overline{V} \cap i(L_n) = (0)$ because C acts as $\frac{n(n+2)}{2}$ on L_n and

$$m(m+2) = n(n+2) \iff m = n \text{ or } m+n = -2.$$

- Case $m = n$. $\quad 0 \longrightarrow L_n \xrightarrow{i} V \xrightarrow{\pi} L_n \longrightarrow 0 \quad — (*)$

basis $\{v_j^{(1)} : 0 \leq j \leq n\}$. $\qquad$ basis $\{v_j^{(2)} : 0 \leq j \leq n\}$.

Choose $w_0 \in V$ s.t. $\pi(w_0) = v_0^{(2)}$. As before define

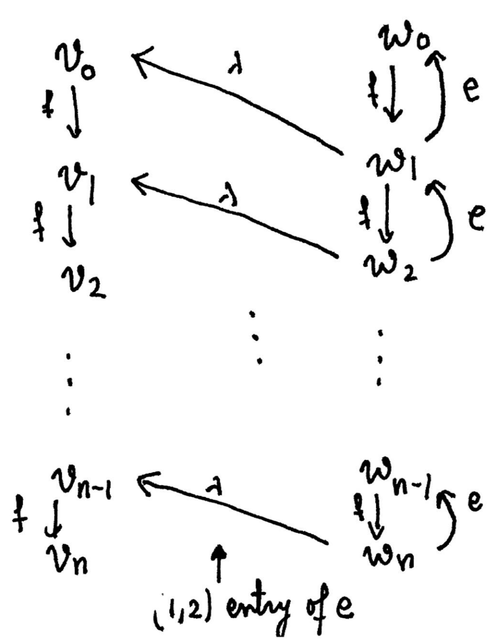
$w_\ell = \frac{f^\ell}{\ell!} w_0$. From $L_n^{(1)}$ we have $v_\ell = i(v_\ell^{(0)})$.

As $(*)$ is a sequence of sl_2-intertwiners, $\exists\, \lambda \in \mathbb{C}$ s.t.

$h.w_0 = n w_0 + \lambda v_0$. A quick calculation then

gives: $\quad h.w_j = (n-2j) w_j + \lambda v_j \quad$ and $\quad e.w_j = (n-j+1) w_{j-1} + \lambda v_{j-1}$

Picture of V:



Claim: $\lambda = 0$ and hence $V \cong L_n \oplus L_n$ as $\mathfrak{sl}_2$-reps.

Proof. $f^{n+1} w_0 = 0 \Rightarrow 0 = e . f^{n+1} w_0$

$$= \underbrace{f^{n+1} \cancel{e w_0}}_{0} + (n+1) f^n (h-n) w_0$$

$$\Rightarrow \quad f^n . (n w_0 + \lambda v_0 - n w_0) = 0 \quad \Rightarrow \quad \lambda . f^n v_0 = 0$$

$f^n v_0 = v_n \neq 0 \quad \Rightarrow \quad \lambda = 0.$ $\square$

6. Corollary. $\mathfrak{sl}_2 \circlearrowright V$ and V finite-dimensional

$\Rightarrow$ h acts semisimply on V; w/ $\mathbb{Z}$-eigenvalues.

<u>Notation and definition.</u> $\forall\, k \in \mathbb{Z}$; $V[k] := \{ v \in V \mid h.v = kv \}$

$\uparrow$ called k-weight space.

$$V = \bigoplus_{k \in \mathbb{Z}} V[k]$$ - called weight space decomposition.

(since h is semisimple)

$\dim V[k] \in \mathbb{Z}_{\geq 0}$ is often called multiplicity (of k in V).

$P(V) := \{ k \in \mathbb{Z} \mid V[k] \neq 0 \}$ $\leftarrow$ weights of V.

(e.g. $P(L_n) = \{ n, n-2, \ldots, -n+2, -n \}$ each weight has multiplicity 1.)

7. Theorem 5 above does not hold if V is infinite-dimensional.

Example. Let $\lambda \in \mathbb{C}$ and consider M_λ to be infinite-dim'l vector space with basis $\{m_0, m_1, m_2, \dots\}$ and sl_2-action:

$$\begin{aligned} h.m_j &= (\lambda - 2j)\, m_j \\ e.m_j &= (\lambda - j + 1)\, m_{j-1} \\ f.m_j &= (j+1)\, m_{j+1} \end{aligned} \qquad (m_{-1} = 0)$$

For $\lambda \notin \mathbb{Z}_{\geq 0}$; M_λ is an irreducible repn. of sl_2.

For $\lambda \in \mathbb{Z}_{\geq 0}$ we have a non-split sequence

$$0 \longrightarrow M_{-\lambda-2} \longrightarrow M_\lambda \longrightarrow L_\lambda \longrightarrow 0$$

f.d. irred. of dim. $\lambda+1$ as before.

[These M_λ's are called Verma modules.]

8. Example of tensor product.

$$L_n \otimes L_1 \cong L_{n+1} \oplus L_{n-1} \qquad \forall\, n \in \mathbb{Z}_{\geq 1}$$

$(L_1 = \mathbb{C}^2 \circlearrowleft sl_2)$

$(L_0 = \mathbb{C}$ trivial repn$)$

basis of $L_1 = \mathbb{C}^2$ is denoted by $\uparrow, \downarrow$

	$v_0 \otimes \downarrow$	$v_1 \otimes \downarrow$		$v_{n-1} \otimes \downarrow$	$v_n \otimes \downarrow$
$v_0 \otimes \uparrow$	$v_1 \otimes \uparrow$	$v_2 \otimes \uparrow$	$\dots$	$v_n \otimes \uparrow$	
$n+1$	$n-1$	$n-3$		$-n+1$	$-n-1$

Weights of $L_n \otimes L_1$

9. Some facts following from our study of $\mathfrak{sl}_2$-representations.

Let $\mathfrak{sl}_2 \circlearrowright V$ be a finite-dimensional repn.

Then
(1) # irreducible constituents of V $=$ $\dim V^0$ where

$$V^0 = \operatorname{Ker}(e) \subseteq V.$$

(2) $$-P(V) = P(V)$$

(3) $$\dim V[k] = \dim V[-k] \quad (\forall k \in \mathbb{Z}.)$$ In fact,

$s = \exp(e)\exp(-f)\exp(e) \in GL(V)$ provides an isomorphism

$$s : V[k] \xrightarrow[\text{as vector spaces}]{\sim} V[-k].$$

[Remark : $V = \mathbb{C}^2$ gives
$$s = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\uparrow \underset{1}{\overset{-1}{\rightleftarrows}} \downarrow$$ action of s.]