Lecture 15

0. Recall - we have been studying the representation theory of $\mathfrak{sl}_2$.

Last time we proved :

(i) Complete reducibility theorem of H. Weyl. - The category of f.d. repns of $\mathfrak{sl}_2(\mathbb{C})$ is semi-simple.

$$\mathrm{Irred}_{f.d.}(\mathfrak{sl}_2(\mathbb{C})) \longleftrightarrow \mathbb{Z}_{\geq 0}$$

set of iso. classes of f.d. irred. repns. of $\mathfrak{sl}_2(\mathbb{C})$

(ii) Let V be a f.d. repn. of $\mathfrak{sl}_2$. We have a naturally defined operator $s \in GL(V)$ given by

$$s = \exp(e)\exp(-f)\exp(e).$$

For each $k \in \mathbb{Z}$; $s : V[k] \xrightarrow{\cong} V[-k]$ vector space iso.

e.g. $V = \mathbb{C}^2$; $s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Remark. The operator s makes sense on any $\mathfrak{sl}_2 \circlearrowright V$, if we only assume that e and f act <u>locally nilpotently</u>.

Definition. Let $X \in \mathrm{End}_{\mathbb{C}}(V)$. We say X acts <u>locally nilpotently on V</u> if $\forall v \in V$, $\exists N > 0$ s.t. $X^N \cdot v = 0$.
(depending on v)

1. Proof that $sl_2 \circlearrowright V$ f.d. ; $v \in V[k] \Rightarrow s(v) \in V[-k]$.

We have to check that $h.(s(v)) = -k\, s(v)$; or equivalently

$$s h s^{-1} = -h \quad (\text{in } \mathrm{End}(V)).$$

The following identity will be useful for this:

$$\boxed{\exp(a)\, b\, \exp(-a) = \exp(\mathrm{ad}(a)) \cdot b}$$

Using this, we get: $s h s^{-1} = \exp(\mathrm{ad}(e)) \exp(-\mathrm{ad} f) \exp(\mathrm{ad}(e)) \cdot h$

which can be computed directly as:

- $$\exp(\mathrm{ad}(e)) \cdot h = h + [e,h] + \frac{\cancel{[e,[e,h]]}^{\,0}}{2!} + \underbrace{\cdots}_{\text{all zero.}}$$
$$= h - 2e$$

- $$\exp(-\mathrm{ad} f) \cdot (h - 2e) = h - [f,h] + \frac{\cancel{[f,[f,h]]}_{\,0}}{2!} + \underbrace{\cdots}_{\text{all zero}}$$
$$-2\left(e - [f,e] + \frac{[f,[f,e]]}{2!} - \frac{\cancel{[f[f,[f,e]]]}_{\,0}}{3!} + \underbrace{\cdots}_{\text{all zero}}\right)$$

$$= h - 2f - 2(e + h - f) = -h - 2e$$

- $\exp(\mathrm{ad}(e)) \cdot (-h-2e)$

$$= -(h-2e) - 2e = -h \quad \text{as claimed.} \qquad \square$$

2. Remark. $\boxed{s_{V_1 \otimes V_2} = s_{V_1} \otimes s_{V_2}}$ and $\boxed{s\Big|_{\mathbb{C}^2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$

can be used to compute the action of s on any irred. L_n $(n \in \mathbb{Z}_{\geq 0})$ as follows.

$V := \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n\text{-fold}}$ - basis given by $\left\{ \underline{a} = a_1 \cdots a_n \text{ where each } a_j \in \{\uparrow, \downarrow\} \right\}$

$\circlearrowright$
s acts by flipping $\begin{array}{l} \uparrow \mapsto (-1) \downarrow \\ \downarrow \mapsto (+1) \uparrow \end{array}$

$$V[n-2k] = \mathrm{Span} \left\{ \underline{a} \;\middle|\; \#\{j : a_j = \downarrow\} = k \right\} \quad \binom{n}{k}\text{-dim'l}$$

We get $\quad s : V[n-2k] \xrightarrow[(-1)^{n-k}]{} V[-n+2k]$

Realizing L_n as subrepn. of V generated by $\uparrow\uparrow\cdots\uparrow$, we get

$$s : v_j \longmapsto (-1)^{n-j} v_{n-j} \qquad \forall\, 0 \leq j \leq n.$$

3. Some more terminology from Lie theory. Again let $\mathfrak{g}$ be a Lie algebra. An <u>ideal</u> in $\mathfrak{g}$ is a subspace $\mathfrak{a} \subseteq \mathfrak{g}$

such that $\left.\begin{array}{l} x \in \mathfrak{g} \\ y \in \mathfrak{a} \end{array}\right\} \Rightarrow [x,y] \in \mathfrak{a}$.

We say $\mathfrak{g}$ is <u>simple</u> if (0) and $\mathfrak{g}$ are the only ideals in $\mathfrak{g}$.

(One dimensional abelian Lie algebra is <u>not considered</u> simple.)

We will now describe how to construct simple Lie algebras (finite-dim'l) starting from an irreducible (finite) root system R.

[c.f. C. Chevalley - Sur la classification des algèbres de Lie simples et de leur représentations (1948)

V. Kac - Infinite dim'l Lie algebras Ch. 1.]

(see Lectures 6,7)

4. Let $R \subset E^* \setminus \{0\}$ be a finite, irreducible, root system.

Assume a fundamental chamber $\mathcal{C}^0$ has been chosen and let $\{\alpha_i\}_{i \in I} \subset R_+$ be the set of simple roots.

<u>Notation</u> $\mathfrak{h} = E \otimes_{\mathbb{R}} \mathbb{C}$. $\nu : \mathfrak{h}^* \longrightarrow \mathfrak{h}$ extending $E^* \xrightarrow{\sim} E$

$$h_i := \alpha_i^\vee = \frac{2\,\nu(\alpha_i)}{(\alpha_i, \alpha_i)} \in \mathfrak{h} \quad (\forall\, i \in I).$$

Definition Let $\tilde{\mathfrak{g}}$ denote the Lie algebra, generated by $\mathfrak{h}$, $\{e_i, f_i\}_{i \in I}$ subject to the following list of relations:

(i) $\mathfrak{h}$ is abelian. That is, $[h, h'] = 0 \ \forall\, h, h' \in \mathfrak{h}$.

(ii) For each $i \in I$, and $h \in \mathfrak{h}$: we have

$$[h, e_i] = \alpha_i(h)\, e_i$$
$$[h, f_i] = -\alpha_i(h)\, f_i$$

(iii) $$[e_i, f_j] = \delta_{ij}\, h_i \quad \forall\, i, j \in I.$$

(recall: $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ - Kronecker's delta.)

Properties of $\tilde{\mathfrak{g}}$: Let $\tilde{n}_\pm$ be the subalgebra of $\tilde{\mathfrak{g}}$, generated by $\{e_i\}_{i\in I}$ (resp. $\{f_i\}_{i \in I}$).

(a) Triangular decomposition $\tilde{\mathfrak{g}} = \tilde{n}_- \oplus \mathfrak{h} \oplus \tilde{n}_+$ (as vector spaces).

(b) $\tilde{n}_\pm$ are freely generated by e_i ($i \in I$) & f_i ($i \in I$) resp.

(c) For $\gamma \in \mathfrak{h}^*$, let $\tilde{\mathfrak{g}}_\gamma := \{ x \in \tilde{\mathfrak{g}} \mid [h, x] = \gamma(h)\, x \ \forall\, h \in \mathfrak{h} \}$

(γ- eigenspace for $\mathfrak{h} \circlearrowright \tilde{\mathfrak{g}}$ via ad)

$$Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\, \alpha_i \subset \mathfrak{h}^*. \quad \text{Then}$$

$$\tilde{n}_\pm = \bigoplus_{\alpha \in Q_+ \setminus \{0\}} \tilde{\mathfrak{g}}_{\pm\alpha} \quad \text{(as vector spaces)}$$

$$\dim \tilde{\mathfrak{g}}_{\pm\alpha} < \infty$$

(d) We have an involution $\tilde{\omega}$ of $\tilde{\mathfrak{g}}$, given on generators

by $\begin{array}{l} e_i \mapsto -f_i \\ f_i \mapsto -e_i \end{array}$ $\forall i \in I$ and $h \mapsto -h \ \forall h \in \mathfrak{h}$.

Remark.- These properties are very easy to prove, especially if we define a "large" representation of $\tilde{\mathfrak{g}}$: $V = \mathbb{C}\langle y_i : i \in I \rangle$

(algebra of non-commuting poly-s in variables y_i $(i \in I)$).

Fix $\lambda \in \mathfrak{h}^*$ and define

$\tilde{\mathfrak{g}} \circlearrowright V$ by :

- f_i = left mult. by y_i $\forall i \in I$.
- $h \cdot 1 = \lambda(h) \cdot 1$ and inductively as:
  $h \cdot (y_j \cdot p) = -\alpha_j(h)\, y_j \cdot p + y_j \cdot h(p)$
- $e_i \cdot 1 = 0$ $\forall i \in I$ and inductively as,
  $e_i \cdot (y_j \cdot p) = \delta_{ij}\, h_i(p) + y_j \cdot e_i(p)$

5. Prop. Let $\mathfrak{a} \subset \tilde{\mathfrak{g}}$ be an ideal. If $\mathfrak{a} \cap \mathfrak{h} \neq (0)$, then $\mathfrak{a} = \tilde{\mathfrak{g}}$.

Proof. If we have $h \in \mathfrak{a} \cap \mathfrak{h}$; $h \neq 0$, then pick $i \in I$ s.t. $\alpha_i(h) \neq 0$.

$$\left.\begin{array}{l} [h, e_i] = \alpha_i(h)\, e_i \\ [h, f_i] = -\alpha_i(h)\, f_i \end{array}\right\} \Rightarrow e_i, f_i \in \mathfrak{a}. \Rightarrow h_i \in \mathfrak{a}.$$

$\Rightarrow$ for $j \in I$ s.t. $a_{ij} \neq 0$; $e_j, f_j, h_j \in \mathfrak{a}$.

Since the Dynkin diagram is connected, we get $e_i, f_i, h_i \in \mathfrak{a}$ $\forall i \in I$
$\Rightarrow \mathfrak{a} = \mathfrak{g}$. $\square$

6. As a consequence of the proposition above, there exists a unique maximal proper ideal $\tilde{\mathfrak{r}} \subsetneq \tilde{\mathfrak{g}}$.

Defn. $\mathfrak{g} := \tilde{\mathfrak{g}} / \tilde{\mathfrak{r}}$ – simple Lie algebra associated to the root system R. (The fact that this is a simple Lie algebra is automatic!)

Remarks. – (i) We don't yet have finite-dimensionality of $\mathfrak{g}$. It will be proved in the next lecture

(ii) We will prove (next lecture) that $\tilde{\mathfrak{r}}$ is generated by

$$\{\theta_{ij}^{\pm}\}_{i,j\in I;\, i\neq j} : \quad \theta_{ij}^{+} = \mathrm{ad}(e_i)^{1-a_{ij}} e_j$$

$$\theta_{ij}^{-} = \mathrm{ad}(f_i)^{1-a_{ij}} f_j .$$