Lecture 16

0. Recall: last time we defined a simple Lie algebra associated to a root system in the following way:

R : finite, irreducible root system $\rightsquigarrow$ $\tilde{\mathfrak{g}}$: Gen-s: $h \in \mathfrak{h}$; $\{e_i, f_i\}_{i \in I}$

Rel^{n}s:

(1) $[h_1, h_2] = 0 \quad \forall h_1, h_2 \in \mathfrak{h}$.

(2) $[h, e_i] = \alpha_i(h) e_i$, $[h, f_i] = -\alpha_i(h) f_i$ $\quad \forall h \in \mathfrak{h}; i \in I$.

(3) $[e_i, f_j] = \delta_{ij} h_i \quad \forall i, j \in I$.

- $R = R_+ \sqcup R_-$
- $\{\alpha_i\}_{i \in I} \subset R_+$ simple roots
- $\nu : E^* \longrightarrow E$, $\alpha_i \longmapsto \nu(\alpha_i)$; $h_i = \alpha_i^\vee = \dfrac{2\nu(\alpha_i)}{(\alpha_i, \alpha_i)}$
- $\mathfrak{h} = E \otimes_{\mathbb{R}} \mathbb{C}$.

$\boxed{\exists! \text{ max'l proper ideal } \tilde{\mathfrak{r}} \subsetneq \tilde{\mathfrak{g}}}$

$\rightsquigarrow$

$$\mathfrak{g} = \tilde{\mathfrak{g}} / \tilde{\mathfrak{r}}.$$

1. Notations regarding $\tilde{\mathfrak{g}}$. $\quad \tilde{\mathfrak{n}}_\pm \subset \tilde{\mathfrak{g}}$ (gen. by $\{e_i\}$; $\{f_i\}$)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad +\qquad -$

$$\tilde{\mathfrak{n}}_\pm = \bigoplus_{\alpha \in Q_+ \setminus \{0\}} \tilde{\mathfrak{g}}_{\pm\alpha} \quad ; \quad Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \subset \mathfrak{h}^*$$

where $\tilde{\mathfrak{g}}_\beta := \{ x \in \tilde{\mathfrak{g}} \mid [h, x] = \beta(h) \cdot x \ \forall h \in \mathfrak{h} \}$ $(\beta \in \mathfrak{h}^*)$.

- $\tilde{\mathfrak{g}}_{k\alpha} = (0)$ for $k \in \mathbb{Z}$, $|k| \geq 2$.

- $\dim \tilde{\mathfrak{g}}_\alpha = \dim \tilde{\mathfrak{g}}_{-\alpha} < \infty$. $\quad \tilde{\mathfrak{g}}_{\pm\alpha_i} = \begin{cases} \mathbb{C} e_i ; + \\ \mathbb{C} f_i ; - \end{cases}$.

For $\beta \in Q_+ \setminus \{0\}$; $ht(\beta) := \sum_{i\in I} n_i$; if $\beta = \sum_{i\in I} n_i\alpha_i$.

(height of β).

2. Remarks about ideals in $\tilde{\mathfrak{g}}$.

Let $\mathfrak{a} \subsetneq \tilde{\mathfrak{g}}$ be a proper ideal. Last time we proved that $\mathfrak{h} \cap \mathfrak{a} = (0)$. Using $\operatorname{ad} f_i \cdot \tilde{\mathfrak{g}}_\beta \subset \tilde{\mathfrak{g}}_{\beta-\alpha_i}$ we can conclude that $\mathfrak{a} \cap \left(\bigoplus_{i\in I} \tilde{\mathfrak{g}}_{\pm\alpha_i}\right) = 0$. Now take $\mathfrak{a}_+ = \mathfrak{a} \cap \tilde{\mathfrak{n}}_+$.

+ or −

Using $\operatorname{ad}(\mathfrak{h})$ - commutation rel^n ($[h, x_\beta] = \beta(h)x_\beta \;\forall\, x_\beta \in \tilde{\mathfrak{g}}_\beta$.)

we conclude that $\mathfrak{a}$ is generated by homogeneous elements:

$$\mathfrak{a}_+ = \bigoplus_{\alpha\in Q_+\setminus\{0\}} \mathfrak{a} \cap \tilde{\mathfrak{g}}_\alpha = \bigoplus_{\alpha\in Q_+\setminus\{0\}} \mathfrak{a}_\alpha \;;\quad \mathfrak{a}_- = \bigoplus_{\alpha\in Q_+\setminus\{0\}} \mathfrak{a}_{-\alpha}.$$

(reason: $\sum_{i=1}^N x_{\beta_i} \in \mathfrak{a}_+ \underset{\substack{\text{(act by}\\ [h,\cdot]\\ \ell\text{-times)}}}{\Rightarrow} \sum_{i=1}^N \beta_i(h)^\ell x_{\beta_i} \in \mathfrak{a}_+ \;\forall\, \ell \geq 0 \Rightarrow x_{\beta_i} \in \mathfrak{a}_+ \;\forall\, i = 1, \dots, N$.)

Lemma. $\mathfrak{a}_\pm$ are ideals in $\tilde{\mathfrak{g}}$. Thus, $\mathfrak{a} = \mathfrak{a}_- \oplus \mathfrak{a}_+$ is a direct sum of ideals.

Proof. $\mathfrak{a}_+$ is clearly stable under $\operatorname{ad}(e_i)$ ($i \in I$) and $\operatorname{ad}(h)$ ($h \in \mathfrak{h}$). Now for each $i \in I$,

$$\mathrm{ad}(f_i)\cdot \mathfrak{a}_+ \subset \bigoplus_{\alpha\in Q_+\setminus\{0;\alpha_i\}_{i\in I}} \mathfrak{a}_{\alpha-\alpha_i} \subset \mathfrak{a}_+ .$$

The argument for $\mathfrak{a}_-$ is similar. □

Cor. If $\alpha \in Q_+\setminus\{0\}$ is minimal (of smallest height for instance) s.t. $\mathfrak{a}_\alpha \neq (0)$; then $\mathrm{ad}(f_i)\cdot\mathfrak{a}_\alpha = 0$ $\forall i\in I$.

3. Conversely, assume given $\theta\in\tilde{\mathfrak{g}}_\alpha$ for some $\alpha\in Q_+\setminus 0$ s.t. $[f_i,\theta]=0$; $\forall i\in I$. Then the ideal generated by θ,

$$\mathrm{ad}(\tilde{\mathfrak{g}})\cdot\theta \subsetneq \tilde{\mathfrak{n}}_+ .$$

Recall: $\tilde{\mathfrak{r}}\subsetneq\tilde{\mathfrak{g}}$ is the unique max'l proper ideal of $\tilde{\mathfrak{g}}$.

By our remarks from §2 above,

$$\tilde{\mathfrak{r}} = \tilde{\mathfrak{r}}_-\oplus\tilde{\mathfrak{r}}_+ \quad ; \quad \tilde{\mathfrak{r}}_\pm = \tilde{\mathfrak{r}}\cap\tilde{\mathfrak{n}}_\pm \text{ are ideals.}$$

(direct sum of ideals)

Let $i\neq j\in I$, and define

$$\theta_{ij}^+ = \mathrm{ad}(e_i)^{1-a_{ij}}\, e_j$$

$$\theta_{ij}^- = \mathrm{ad}(f_i)^{1-a_{ij}}\, f_j .$$

Lemma. $\forall k\in I$,

$$\mathrm{ad}(e_k)\cdot\theta_{ij}^- = 0$$

$$\mathrm{ad}(f_k)\cdot\theta_{ij}^+ = 0 .$$

Hence $\theta_{ij}^\pm\in\tilde{\mathfrak{r}}_\pm$.

Proof. (for − case.) $\theta_{ij}^- = (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j$.

- $k \neq i$ or j. $[e_k, \theta_{ij}^-] = 0$ since $[e_k, f_i] = 0 = [e_k, f_j]$.
- $k = j$. $[e_j, (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j] = (\mathrm{ad}\, f_i)^{1-a_{ij}} h_j$
  - $a_{ij} < 0$: this commutator is 0 since $[f_i, f_i] = 0$
  - $a_{ij} = 0$: $[f_i, h_j] = \alpha_i(h_j) f_i = a_{ji} f_i = 0$.
    (so $a_{ji} = 0$ as well)
- $k = i$. This is $\mathfrak{sl}_2$ - computation again.

$$\left.\begin{aligned}(\mathrm{ad}\, e_i)\cdot f_j &= 0\\ (\mathrm{ad}\, h_i)\cdot f_j &= -a_{ij} f_j\end{aligned}\right\} \Rightarrow \mathrm{ad}(e_i)\cdot\left(\frac{\mathrm{ad}(f_i)^{\ell}}{\ell!}\cdot f_j\right)$$

$$= (-a_{ij} - \ell + 1)\left(\frac{(\mathrm{ad}\, f_i)^{\ell-1}}{(\ell-1)!}\cdot f_j\right)$$

$$= 0 \quad \text{for } \ell = 1 - a_{ij}.$$

□

4. Consequence of Lemma 3 − Weyl group action.

Let $\bar{\mathfrak{g}} = \tilde{\mathfrak{g}} \big/ \langle \theta_{ij}^{\pm} : i \neq j \in I \rangle = \underbrace{\bar{\mathfrak{n}}_-}_{\tilde{\mathfrak{n}}_- / \langle \theta_{ij}^- \rangle} \oplus \mathfrak{h} \oplus \underbrace{\bar{\mathfrak{n}}_+}_{\tilde{\mathfrak{n}}_+ / \langle \theta_{ij}^+ \rangle}$

For each $i \in I$; $\underset{\underset{\mathbb{C}\text{-span of } \{h_i, e_i, f_i\}}{\|}}{sl_2^{(i)}} \underset{\text{adjoint-action}}{\curvearrowright} \overline{\mathfrak{g}}$ is $\underline{\text{locally}}$ (lecture 15 page 2)

$\underline{\text{nilpotent}}$ because of relations $\{\theta^+_{jk} \mid j \neq k \in I\}$. Hence we obtain

$s_i \in \underset{\text{Lie-alg}}{\mathrm{Aut}} (\overline{\mathfrak{g}})$. An easy computation shows that (Lecture 15, page 2)

$$s_i : \overline{\mathfrak{g}}_\alpha \longrightarrow \overline{\mathfrak{g}}_{s_i(\alpha)} \quad \text{vector space iso.}$$

Corollary. - (i) $\overline{\mathfrak{g}}_\alpha \neq 0 \Rightarrow \overline{\mathfrak{g}}_{w(\alpha)} \neq 0$ (& of same dim)

$\forall w \in W \leftarrow$ Weyl group of R.

(ii) $\overline{\mathfrak{n}}_\pm = \tilde{\mathfrak{n}}_\pm / \langle \theta^\pm_{ij} \rangle \longrightarrow\!\!\!\!\rightarrow \tilde{\mathfrak{n}}_\pm / \tilde{\mathfrak{r}}_\pm$ is an iso.

Proof of (ii) (+ case) Let $k = \mathrm{Ker}(\pi_+) \cong \tilde{\mathfrak{r}}_+ / \langle \theta^+_{ij} \rangle \subset \overline{\mathfrak{n}}_+$.

If $k \neq (0)$, let $\alpha \in Q_+ \setminus \{0\}$ be of minimum height s.t. $k_\alpha \neq (0)$

Since we have $\underset{(i \in I)}{s_i}$ acting on $\overline{\mathfrak{g}}$, and $x \in \mathfrak{a} \subsetneq \overline{\mathfrak{g}} \Rightarrow s_i \cdot x \in \mathfrak{a}$ (ideal, if exists)

(as $s_i \cdot x$ is expressed in terms of $\mathrm{ad}(e_i)$ and $\mathrm{ad}(f_i)$) ; we get

that $k_{s_i(\alpha)} \neq (0) \Rightarrow ht(s_i(\alpha)) \geq ht(\alpha)$ $(\forall i \in I)$

$\Rightarrow (\alpha, \alpha_i) \leq 0$ $(\forall i \in I)$.

Now $\underset{Q_+\setminus\{0\}}{\ni} \alpha = \sum_{i\in I} n_i\alpha_i$; $n_i \in \mathbb{Z}_{\geq 0}$. Thus, we get

$$(\alpha,\alpha) = \sum_{i\in I} n_i \underbrace{\boxed{(\alpha,\alpha_i)}}_{\leq 0 \text{ as on the last page}} \leq 0 \Rightarrow \alpha = 0.$$

This is a contradiction. $\square$

5. The same argument, as in the proof of the Corollary above, proves that $\mathfrak{g}_\alpha \neq 0 \Leftrightarrow \alpha \in R_+$. Thus, if $\alpha = w(\alpha_i)$, then
$(\alpha \in Q_+\setminus\{0\})$

$\mathfrak{g}_\alpha \cong \mathfrak{g}_{\alpha_i} = \mathbb{C}\cdot e_i$ is 1-dimensional

We record these observations as:

<u>Prop.</u> (i) $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$; where $\mathfrak{n}_\pm = \bigoplus_{\alpha\in R_+} \mathfrak{g}_{\pm\alpha}$.

(ii) $\dim \mathfrak{g}_\alpha = 1$ $\forall \alpha \in R$.

Therefore $\dim \mathfrak{g} = |I| + 2|R_+| < \infty$.

Remark. - It is a non-trivial, and beautiful theorem, that every finite-dimensional simple Lie algebra arises this way. (usually covered in Lie Algebras course).