Lecture 17

0. Summary of the last few lectures.

R : finite irreducible root system. $\{\alpha_i\}_{i\in I}$ a choice of base – simple roots

$$R = R_+ \sqcup R_- \ ; \ R_- = -R_+$$

$$\mathfrak{h}^* \xrightarrow[\nu]{\sim} \mathfrak{h} \ ; \ h_i := \frac{\nu(\alpha_i)\cdot 2}{(\alpha_i,\alpha_i)} \qquad \left(\text{Notation } d_\alpha = \frac{(\alpha,\alpha)}{2}\right)$$

$$= \frac{\nu(\alpha_i)}{d_i} \quad \forall i\in I.$$

$$A = \left(a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}\right)_{i,j\in I}.$$

(Cartan Matrix)

W = Weyl group, generated by $\{s_i\}_{i\in I}$

$[s_i^2 = 1 ; (s_is_j)^{m_{ij}} = 1$

$a_{ij}a_{ji}$	0	1	2	3
m_{ij}	2	3	4	6

.]

We defined a Lie algebra

$\mathfrak{g}(A)$: generators $\{e_i, f_i, h_i\}_{i\in I}$

relations :

(1) $[h_i, h_j] = 0 \quad \forall i,j\in I.$

(2) $[h_i, e_j] = a_{ij}e_j \ ; \ [h_i, f_j] = -a_{ij}f_j$

(3) $[e_i, f_j] = \delta_{ij}h_i \quad \forall i,j\in I$

(4) $\forall i\neq j\in I ; \ (\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0 = (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j.$

1. Remarks.

$\mathfrak{h} = \mathbb{C}$-span of $\{h_i : i\in I\}$. (1) $\Leftrightarrow$ $\mathfrak{h}$ is an abelian Lie algebra : $[h,h'] = 0 \ \forall h,h'\in\mathfrak{h}$.

(2) : $\mathfrak{g} \circlearrowleft \mathfrak{h}$ via ad is semisimple $\rightsquigarrow$ simultaneous eigenspaces

$$\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i \ \ \& \ \ \mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i.$$

i.e. $[h, e_j] = \alpha_j(h) e_j$ and $[h, f_j] = -\alpha_j(h) f_j$

(recall: $\alpha_j(h_i) = \frac{1}{d_i}(\alpha_i, \alpha_j) = a_{ij} \quad \forall i, j \in I.$)

(3) Every $i \in I$ gives a copy of $\mathfrak{sl}_2$ - $\{e_i, f_i, h_i\}$; denoted by $\mathfrak{sl}_2^{(i)} \subset \mathfrak{g}$.

(4) For every $i \neq j \in I$; $\mathfrak{sl}_2^{(i)} \circlearrowright \bigoplus_{k \geq 0} \mathfrak{g}_{\alpha_j + k\alpha_i}$

Then this is an irreducible repn. of $\mathfrak{sl}_2^{(i)} \cong L_{-a_{ij}}$ (recall $a_{ij} \in \mathbb{Z}_{\leq 0}$)

This allows us to compute the action of $\{\bar{s}_i\}_{i \in I} \subset \mathrm{Aut}_{\text{Lie-alg}}(\mathfrak{g})$

(recall $\bar{s}_i = \exp(\mathrm{ad}\, e_i) \exp(-\mathrm{ad}\, f_i) \exp(\mathrm{ad}\, e_i)$
$\in \mathrm{Aut}(\mathfrak{g}) \rightsquigarrow \bar{s}_i : \mathfrak{g}_\alpha \xrightarrow{\sim} \mathfrak{g}_{s_i(\alpha)}$)

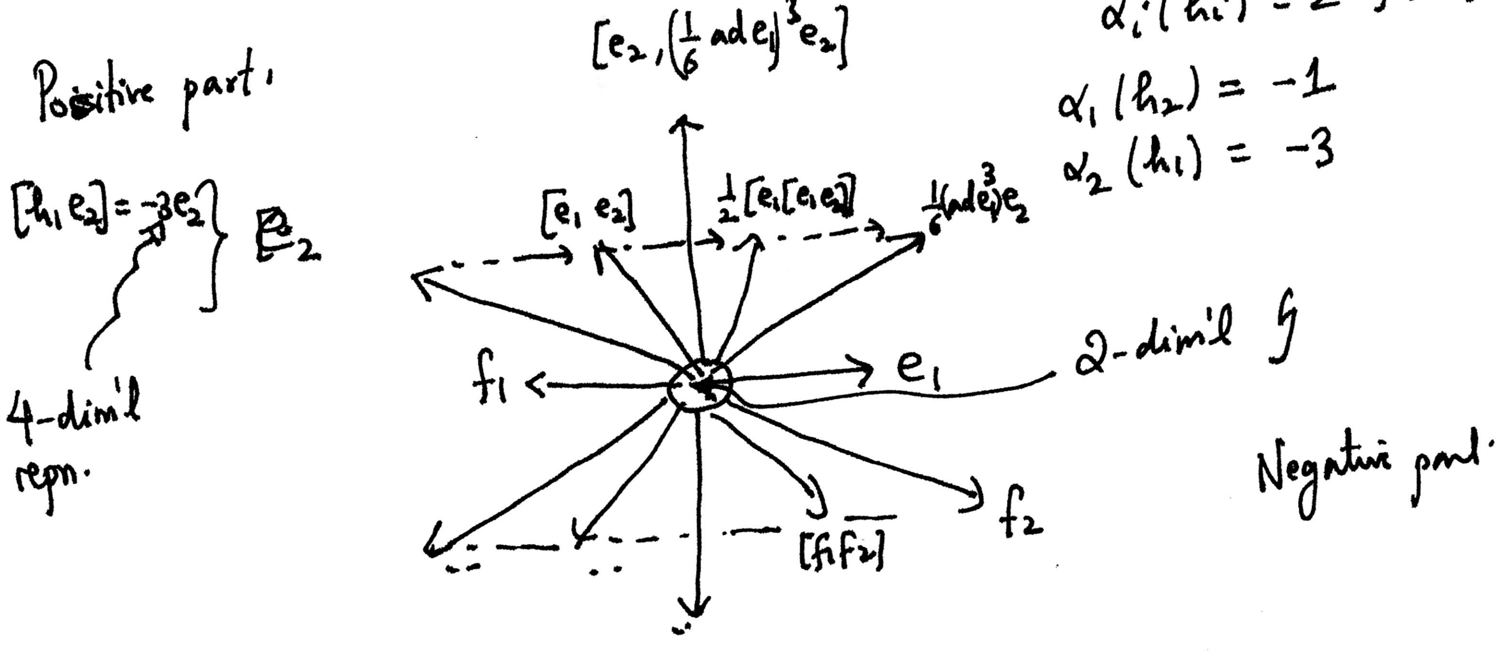
2. Example G_2. $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. $\mathfrak{h} = \mathbb{C} h_1 \oplus \mathbb{C} h_2$

$\mathfrak{h}^*$ - basis α_1, α_2 s.t.

$\alpha_i(h_i) = 2$; $i = 1, 2$

$\alpha_1(h_2) = -1$

$\alpha_2(h_1) = -3$



3. $(\cdot,\cdot)$ and Casimir element.

$\mathfrak{g}$ has a unique, non-degenerate, symmetric, bilinear form $(\cdot,\cdot): \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ s.t.

(i) $(\cdot,\cdot)\big|_{\mathfrak{h}\times\mathfrak{h}}$ is given by the original inner product on $\mathfrak{h}(= E\otimes_{\mathbb{R}}\mathbb{C})$.

(ii) $\forall\, i,j\in I;\quad (e_i, f_j) = \dfrac{\delta_{ij}}{d_i}$

(So, e_i is dual to $d_i f_i$.)

(iii) Invariance property: $\forall\, x,y,z\in\mathfrak{g}$,
$$([x,y],z) = (x,[y,z]).$$

Properties. - • $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are dual to each other.
• $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha+\beta\neq 0$. (hint: use (iii) $y=h\in\mathfrak{h}$.)

Definition. Pick a basis (o.n.) of $\mathfrak{h}$: $\{x_i\}_{i\in I}$ (s.t. $(x_i,x_j)=\delta_{ij}$).
$$C^0 := \sum_{i\in I} x_i^2 \quad \text{(on a representation, say } V\text{)}$$

$K_\alpha := d_\alpha(e_\alpha f_\alpha + f_\alpha e_\alpha)$ where $e_\alpha\in\mathfrak{g}_\alpha$ and $f_\alpha\in\mathfrak{g}_{-\alpha}$ are chosen so that $(e_\alpha,f_\alpha)=\frac{1}{d_\alpha}$.

Casimir element. Let $\mathfrak{g} \circlearrowright V$ be a representation

$$C_V := C_V^0 + \sum_{\alpha \in R_+} K_{\alpha;V}$$

$$= \sum_{i \in I} x_i^2 + \sum_{\alpha \in R_+} d_\alpha (e_\alpha f_\alpha + f_\alpha e_\alpha)$$

Similarly we will define

$$\Omega = \underbrace{\boxed{\sum_{i \in I} x_i \otimes x_i}}_{\Omega^0} + \sum_{\alpha \in R_+} d_\alpha (e_\alpha \otimes f_\alpha + f_\alpha \otimes e_\alpha) \in \mathfrak{g} \otimes \mathfrak{g}.$$

This is nothing but <u>the canonical tensor of</u> $(\cdot,\cdot)$ on $\mathfrak{g}$.

(If V is a f.d. vector space / $\mathbb{C}$ and $(\cdot,\cdot) : V \times V \to \mathbb{C}$ a non-degenerate, symmetric bilinear form, we can define a natural element of $V \otimes V$ - pick a basis of V: $\{v_i\}_i$.

$\{v^i\} :=$ dual basis

canonical tensor of $(\cdot,\cdot)$ $\to$ $\Omega := \sum_i v^i \otimes v_i$

depends only on $(\cdot,\cdot)$ and not the choice of a basis.)

Let $\mathfrak{g} \circlearrowright V$ be a repn.

- For any $x \in \mathfrak{g}$; we get

$$\left.\begin{aligned} [x \otimes 1 + 1 \otimes x, \Omega_V] &= 0. \\ [x, C_V] &= 0. \end{aligned}\right\} \text{ we will see a proof of this later. (next lecture).}$$

4. Finite-dimensional representations of $\mathfrak{g}$.

Let V be a f.d. vector space / $\mathbb{C}$; and $\mathfrak{g} \circlearrowright V$.

Prop. $\mathfrak{h} \circlearrowright V$ is diagonalizable

Proof $\forall\, i \in I$; $sl_2^{(i)} \circlearrowright V \Rightarrow h_i$ is diagonalizable

$[h_i, h_j] = 0$ (a set of commuting; diagonalizable matrices can be simultaneously diagonalized.)

$\Rightarrow \mathfrak{h} \circlearrowright V$ is diagonalizable. $\square$

called "μ-weight space" of V

$\forall\, \mu \in \mathfrak{h}^*$, define $V[\mu] := \{ v \in V \mid h \cdot v = \mu(h) v \ \forall h \in \mathfrak{h} \}$.

Thus, by Prop. above, $V = \bigoplus_{\mu \in \mathfrak{h}^*} V[\mu]$.

5. Weight lattice. $\mathfrak{g} \circlearrowright V$ (f.d. repn). $sl_2^{(i)} \circlearrowright V$

implies $V[\mu] \neq 0 \Rightarrow \mu(h_i) \in \mathbb{Z} \ \forall\, i \in I$.

Define $\underset{\text{(weight lattice)}}{P} := \{ \mu \in \mathfrak{h}^* \mid \mu(h_i) \in \mathbb{Z} \ \forall\, i \in I \} \subset \mathfrak{h}^*$.

Thus $V = \bigoplus_{\mu \in P} V[\mu]$. $\underset{\text{set of weights of } V.}{P(V) := \{ \mu \in P \mid V[\mu] \neq 0 \}}$

6. Now we assume that $\mathfrak{g} \circlearrowright V$ is an <u>irreducible</u> finite-dim'l representation.

Proposition.~ There is $\lambda \in P_+$ (i.e. $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ $\forall i \in I$.) and a unique up to scalar vector $v_\lambda \in V$ s.t. $h v_\lambda = \lambda(h) v_\lambda$ $\forall h \in \mathfrak{h}$
i.e. $v_\lambda \in V[\lambda]$ $e_i v_\lambda = 0$ $\forall i \in I$.

Then, $V[\lambda]$ is 1-dim'l ; and $\forall i \in I$

$$f_i^{\lambda(h_i)+1} v_\lambda = 0$$

<u>Proof</u> Pick $\lambda \in P$ s.t. $V[\lambda] \neq 0$ & $V[\lambda+\alpha_i] = 0$ $\forall i \in I$
(exists by finite-dimensionality)

Let $v \in V[\lambda]$; $v \neq 0$. Then $h \cdot v = \lambda(h) v$ $\forall h \in \mathfrak{h}$; and
$e_i v \in V[\lambda+\alpha_i] = (0)$. $\forall i \in I$.

V' = subrepn. generated by v = $\mathbb{C}$-span of $\{f_{i_1} \cdots f_{i_r} v\}_{\substack{r \geq 0 \\ i_1, \dots, i_r \in I}}$
= V by irreducibility
($V' \neq (0)$ since, for $r=0$, $v \in V'$).

$sl_2^{(i)} \circlearrowright V$ and $h_i v_\lambda = \lambda(h_i) v_\lambda$ $\Rightarrow$ $f_i^{\lambda(h_i)+1} v_\lambda = 0$

□

7. Conversely, starting from $\lambda \in P_+$, we can define

L_λ = representation of $\mathfrak{g}$ generated by a (non-zero) vector v_λ, subject to the following list of relations:

$$\left.\begin{aligned} h \cdot v_\lambda &= \lambda(h)\, v_\lambda \quad \forall\, h \in \mathfrak{h}. \\ e_i \cdot v_\lambda &= 0 \quad \forall\, i \in I \end{aligned}\right\} \quad (*)$$

$$\boxed{f_i^{\lambda(h_i)+1}\, v_\lambda = 0 \quad \forall\, i \in I} \; - (**)$$

Theorem (Harish-Chandra) L_λ is a finite-dimensional irreducible representation of $\mathfrak{g}$.

Idea of the proof of finite-dimensionality:

- use $(**)$ to prove that e_i, f_i act locally nilpotently on L_λ; $\forall\, i \in I$. This gives $\overline{s_i} \in GL(L_\lambda)$ s.t.

  $$\forall\, \mu \in P(L_\lambda)\ ; \quad \overline{s_i} : L_\lambda[\mu] \to L_\lambda[s_i \mu].$$

  Hence $P(L_\lambda) \subset P$ is W-stable subset.

- Use $(*)$ to prove that $P(L_\lambda) \subset \lambda - Q_+$ where

  $$Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\, \alpha_i$$

- Since W acts transitively on the set of chambers we conclude that $|P(L_\lambda)| < \infty$; hence $\dim L_\lambda = \sum_{\mu \in P(L_\lambda)} \dim L_\lambda[\mu] < \infty$. $\square$

8. Characters. $\mathfrak{g} \circlearrowright V$ (f.d.) $\Rightarrow V = \bigoplus_{\mu\in P} V[\mu]$.

Define $$\chi_V = \sum_{\mu\in P} (\dim V[\mu])\, e^{\mu} \in \mathbb{Z}[e^{\pm\omega_i} : i\in I]$$
(ring of Laurent poly-s in $|I|$ variables).

Properties

(i) $\chi_{V_1\oplus V_2} = \chi_{V_1} + \chi_{V_2}$

(ii) $\chi_{V_1\otimes V_2} = \chi_{V_1}\cdot\chi_{V_2}$

$\mathrm{Rep}_{fd}(\mathfrak{g})$ is a semisimple category – i.e., every f.d. ~~irreducible~~ repn. of $\mathfrak{g}$ is a direct sum of irreducible ones.

Character formula of Hermann Weyl:

$$\boxed{\chi_{L_\lambda} = \frac{\sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R_+} 1-e^{-\alpha}}}$$
; where

$\rho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$ is the unique element of P_+ s.t. $\rho(h_i)=1$ $\forall i\in I$.

9. Example : $\mathfrak{g} = \mathfrak{sl}_2 \circlearrowright L_n$.

Direct computation : L_n has a basis $\{v_0, v_1, \dots, v_n\}$ and $h.v_j = (n-2j)v_j$

$$\Rightarrow \chi_{L_n} = e^{n\frac{\alpha}{2}} + e^{(n-2)\frac{\alpha}{2}} + \dots + e^{-n\frac{\alpha}{2}}$$

$$= z^n + z^{n-2} + \dots + z^{-n} \quad \text{where } z = e^{\alpha/2}$$

Weyl Character formula,

$$\chi_{L_n} = \frac{e^{n\frac{\alpha}{2}} - e^{(-n-2)\frac{\alpha}{2}}}{1 - e^{-\alpha}} = \frac{z^n - z^{-n-2}}{1 - z^{-2}}$$

$$= \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} = z^n + z^{n-2} + \dots + z^{-n} .$$

$\square$