Lecture 18

Knizhnik - Zamolodchikov equations.

0. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, and assume that we have a non-degenerate, symmetric bilinear form $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ such that

$$([x,y], z) = (x, [y,z]) \quad \forall\, x,y,z \in \mathfrak{g}. \qquad -(0.1)$$

A bilinear form satisfying (0.1) is called <u>invariant</u> form on $\mathfrak{g}$.

e.g. $\mathfrak{g} = \mathfrak{gl}_m(\mathbb{C})$ (Lie algebra of $m \times m$ matrices over $\mathbb{C}$)

$(x,y) :=$ Trace (xy) is an example of invariant, non-deg symm. bilinear form

1. A Lie algebra together with a symm. non-deg invariant bilinear form is called a <u>quadratic</u> Lie algebra.

Definition. The <u>Casimir tensor</u> $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the canonical tensor of $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$.

That is, let $\{x_i\}$ be a basis of $\mathfrak{g}$ and $\{y_i\}$ be the dual basis.

Then $\Omega = \sum x_i \otimes y_i$.

Lemma. - For every $x \in \mathfrak{g}$ we have

$$[x \otimes 1 + 1 \otimes x, \Omega] = 0.$$

Proof. Let $[x, x_i] = \sum_j c_{ij} x_j$ and $[x, y_i] = \sum_j d_{ij} y_j$

Then $[x \otimes 1 + 1 \otimes x, \Omega] = \sum_i [x, x_i] \otimes y_i + x_i \otimes [x, y_i]$

Coefficient of $x_k \otimes y_\ell$ on the right-hand side is then

$$c_{\ell k} + d_{k\ell} = \left([x, x_\ell], y_k\right) + \left(x_\ell, [x, y_k]\right)$$
$$= 0 \text{ by invariance.}$$

$\square$

2. Again let $\mathfrak{g}$ be a quadratic Lie algebra and $n \in \mathbb{Z}_{\geq 2}$.

Let $V_1, \ldots, V_n$ be f.d. representations of $\mathfrak{g}$.

$F := V_1 \otimes \cdots \otimes V_n$ and $\forall\, i < j$ define $\Omega_{ij} \in \mathrm{End}(F)$

If $\Omega = \sum_a x_a \otimes x^a$ (here $\{x_a\}$ is a basis of $\mathfrak{g}$ and $\{x^a\}$ is the dual basis)

then $\Omega_{ij} := \underbrace{\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}}_{\substack{i-1 \\ \text{terms}}} \otimes \underset{\uparrow \atop i^{\text{th}} \text{ spot}}{x_a} \otimes \underbrace{\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}}_{\substack{j-i-1 \\ \text{terms}}} \otimes \underset{\uparrow \atop j^{\text{th}} \text{ spot}}{x^a} \otimes \underbrace{\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}}_{\substack{n-j \\ \text{terms}}}$

Remark. $\Omega = \sum_a x^a \otimes x_a$ (by independence from the choice of a basis)

$= \Omega_{21}$

$\rightsquigarrow \Omega_{ij} = \Omega_{ji}$.

KZ connection $\boxed{\nabla = d - \varkappa \sum_{i<j} \frac{d(z_i - z_j)}{z_i - z_j}\, \Omega_{ij}}$ $\varkappa \in \mathbb{C}$ is arbitrary.

Kappa

That is, for a function $f(z_1, \dots, z_n)$ of n complex variables (valued in F or $GL(F)$) ; KZ equations are the following system of PDE's.

$$\boxed{\frac{\partial f}{\partial z_i} = \varkappa \sum_{\substack{j \neq i \\ 1 \leq j \leq n}} \frac{\Omega_{ij}}{z_i - z_j} f} \qquad (1 \leq i \leq n).$$

Proposition. - (1) KZ equations are consistent.

(2) If $V_1 = \dots = V_n = V$, then $S_n \circlearrowright F = V^{\otimes n}$; and KZ equations are S_n - equivariant.

Proof. (1) By Kohno's Lemma (lecture 4, §3, page 6) we need to prove that

$$\begin{cases} [\Omega_{ij}, \Omega_{k\ell}] = 0 & \text{for } i, j, k, \ell \text{ distinct} \\ [\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] = 0 & \text{for } i, j, k \text{ distinct} \end{cases}$$

The first relation is obvious since Id commutes with any operator. The second can be written - for $i, j, k = 1, 3, 2$ -

$$[\Omega_{13} + \Omega_{23}, \Omega_{12}] = 0$$

$\Omega = \sum x_a \otimes x^a \Rightarrow$ LHS of the desired eq$^{\underline{n}}$ is $\sum_a [x_a \otimes 1 + 1 \otimes x_a, \Omega] \otimes x^a$

$= 0$ by Lemma 1 above.

(2) follows from $\sigma\,\Omega_{ij}\,\sigma^{-1} = \Omega_{\sigma(i),\sigma(j)}$ $\quad \forall\, \sigma \in S_n,\ i \neq j \in \{1,\dots,n\}$.

$\square$

3. Example. $n = 2$.

$$\frac{\partial f}{\partial z_1} = \frac{\varkappa\,\Omega\, f}{z_1 - z_2} \quad ; \quad \frac{\partial f}{\partial z_2} = \frac{-\varkappa\,\Omega}{z_1 - z_2}\, f$$

$\Rightarrow\ f = (z_1 - z_2)^{\varkappa\Omega}$ and the monodromy $\mu_f\big(z_1 \circlearrowright z_2\big) = e^{2\pi i \varkappa \Omega}$

full loop.

$n = 3$.

$$\left.\begin{aligned}
\frac{\partial f}{\partial z_1} &= \varkappa\left(\frac{\Omega_{12}}{z_1 - z_2} + \frac{\Omega_{13}}{z_1 - z_3}\right) f \\
\frac{\partial f}{\partial z_2} &= \varkappa\left(-\frac{\Omega_{12}}{z_1 - z_2} + \frac{\Omega_{23}}{z_2 - z_3}\right) f \\
\frac{\partial f}{\partial z_3} &= \varkappa\left(\frac{-\Omega_{13}}{z_1 - z_3} - \frac{\Omega_{23}}{z_2 - z_3}\right) f
\end{aligned}\right]$$

Change of variables

$$z_1 - z_2 = u \cdot z$$
$$z_1 - z_3 = u$$
$$z_2 - z_3 = u(1-z)$$

commutes w/ Ω_{12}, Ω_{23} & Ω_{13}.

$$\Rightarrow\ \nabla = d - \varkappa\left(\frac{du}{u}\cdot\left(\Omega_{12} + \Omega_{23} + \Omega_{13}\right) + \frac{dz}{z}\,\Omega_{12} + \frac{dz}{z-1}\,\Omega_{23}\right)$$

Fundamental solution $f = \tilde{f} \cdot u^{\varkappa(\Omega_{12} + \Omega_{23} + \Omega_{13})}$ where

$$\frac{d\tilde{f}}{dz} = \varkappa\left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z-1}\right)\tilde{f}$$

(Drinfeld ODE w/ $A = \varkappa\,\Omega_{12}$, $B = \varkappa\,\Omega_{23}$.)

KZ associator Φ_{KZ} = Drinfeld associator with substitution

$$A = \varkappa \Omega_{12} \;;\; B = \varkappa \Omega_{23}.$$

In conclusion, we can incorporate S_n-action as follows.

• $(n=2)$ $\mathfrak{g} \curvearrowright V_1, V_2 \rightsquigarrow V_1 \otimes V_2 \xrightarrow{(12)\circ e^{\pi i \varkappa \Omega}} V_2 \otimes V_1$

is a $\mathfrak{g}$-intertwiner, again because

Monodromy of "half loop" of Lemma 1.

$$z_1 \quad z_2 \quad \rightsquigarrow \quad z_2 \quad z_1$$

• $(n=3)$ For $\mathfrak{g} \curvearrowright V_1, V_2, V_3$ we get

$$\Phi_{KZ} \in \mathrm{End}_{\mathfrak{g}}(V_1 \otimes V_2 \otimes V_3) \quad \text{(invertible)}$$

Let $f_0 = \tilde{f}_0 \cdot u^{\varkappa(\Omega_{12}+\Omega_{23}+\Omega_{13})}$ be the fundamental soln.

near $z=0$, i.e. $\tilde{f}_0 = H(z)\cdot z^{\varkappa \Omega_{12}}$.

$$\mu_{f_0}\left(\text{half loop } z_1 \; z_2\right) = (12)\cdot e^{\pi i \varkappa \Omega_{12}} : V_1 \otimes V_2 \otimes V_3 \to V_2 \otimes V_1 \otimes V_3.$$

$$\mu_{f_0}\left(\text{half loop } z_2 \; z_3\right) = \Phi^{-1}_{V_1,V_3,V_2} \cdot (23)\, e^{\pi i \varkappa \Omega_{23}} \cdot \Phi_{V_1,V_2,V_3}$$

4. Drinfeld's main observation is that ∇_{KZ} gives rise to a (highly non-trivial) structure of "braided tensor category" on the category of finite-dim'l representations of $\mathfrak{g}$.

We will prove this result later, after having defined the braided tensor categories.

5. Casimir element and the Casimir connection.

Now let $\mathfrak{g}$ be the simple Lie algebra associated to a (finite, irreducible) root system R.

$$\nabla_C := d - \varkappa \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} K_\alpha \quad ; \text{ where}$$

(C casimir connection)

$\forall \alpha \in R_+$, $K_\alpha = d_\alpha (e_\alpha f_\alpha + f_\alpha e_\alpha)$ - recall $\mathfrak{g}$ has a non-deg, invariant, symmetric, bilinear form - relative to which the root spaces $\mathfrak{g}_\alpha$ & $\mathfrak{g}_{-\alpha}$ are dual to each other. We picked $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ s.t. $(e_\alpha, f_\alpha) = \frac{1}{d_\alpha}$ $\left(\text{recall: } d_\alpha = \frac{(\alpha,\alpha)}{2}\right)$.

Theorem (Millson-Toledano Laredo) ∇_C is flat and W-equivariant

(Weyl group)

In conclusion, for any f.d. repn. $\mathfrak{g} \circlearrowright V$ we have a system of PDE's for a function $f : \mathfrak{h}^{reg} \to GL(V)$

$$\mathfrak{h}^{reg} = \Big(\mathfrak{h} \setminus \bigcup_{\alpha \in R_+} \mathrm{Ker}(\alpha)\Big)$$

$$\boxed{\frac{\partial f}{\partial \alpha_i} = \varkappa \sum_{\alpha \in R_+} \frac{n_i(\alpha)}{\alpha} K_\alpha \cdot f}$$

where $\alpha = \sum_{i \in I} n_i(\alpha)\alpha_i$

$\forall i \in I$.

Remark. W does not act on V - but this can be fixed by defining $\overline{s}_i = \exp(e_i)\exp(-f_i)\exp(e_i) \in GL(V)$ which gives an action of (a finite extn. of W) $\widetilde{W} \circlearrowright V$.

Monodromy of ∇_c thus gives rise to an action of $\pi_1(\mathfrak{h}^{reg}/W)$ $= B_W$ on V.

The structure on $\mathrm{Rep}_{fd}(\mathfrak{g})$ that comes from ∇_c is called that of a "Coxeter category" - (Appel-Toledano Laredo). We may not have time to go over this recent & sophisticated theory -

A. Appel & V. Toledano Laredo - Coxeter categories & quantum groups
Selecta Math. (2019) vol. 25, pages 1-97.