Lecture 19

Braided tensor categories

0. Let $\mathcal{C}$ be a category, abelian and $\mathbb{C}$-linear: this consists of the data of
   - a class of objects
   - for X, Y objects - a $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathcal{C}}(X,Y)$
   - for X, Y, Z objects, a $\mathbb{C}$-bilinear composition

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$$
$$(f, g) \longmapsto g \circ f.$$

such that (i) $\forall X \in \mathcal{C}$, we have a distinguished $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$;
s.t. $\mathrm{Id} \circ f = f$ and $g \circ \mathrm{Id} = g$.

(ii) Composition is associative.

(Abelian) - $\begin{cases} \text{finite direct sums exist.} \\ \text{Kernels and Cokernels exist.} \\ \text{Every bijection is an isomorphism.} \end{cases}$

The only examples we will consider in this course will be that of $\boxed{\mathrm{Rep}_{fd}(A)}$ - here A is an associative, unital algebra$/\mathbb{C}$

$\begin{cases} \text{Objects: representations of } A \text{ - i.e. a f.d.} \\ \quad \mathbb{C}\text{-vector space } V \text{ together with } A \xrightarrow{\pi} \mathrm{End}_{\mathbb{C}}(V) \\ \quad \text{(algebra hom.)} \end{cases}$

$$\mathrm{Hom}_{\mathcal{C}}(V,W) = \mathrm{Hom}_{A}(V,W) = \left\{ \underset{\mathbb{C}\text{-linear}}{f: V \to W} \;\middle|\; \begin{matrix} f(a \cdot v) = a \cdot f(v) \\ \forall a \in A, v \in V \end{matrix} \right\}$$

1. A <u>tensor category</u> (or monoidal category) is a ($\mathbb{C}$-linear...) category $\mathcal{C}$ together with a bifunctor, additive in each component:

- $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

- an associativity constraint a - that is, $\forall\, X, Y, Z \in \mathcal{C}$,

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z) : \text{isomorphism}$$

which is natural in X, Y and Z.

[Thus, $a : \otimes \circ (\otimes \times \mathrm{Id}) \xrightarrow{\sim} \otimes \circ (\mathrm{Id} \times \otimes) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.]

(natural trans. of functors)

- Unit object $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$ together with

$$\forall X \in \mathcal{C} \quad \begin{cases} l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X & \text{(left unit constraint)} \\ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \end{cases}$$

(natural in X.)

Satisfying two axioms:

(Pentagon axiom) For $X_1, X_2, X_3, X_4 \in \mathcal{C}$, the following diagram commutes

$$\begin{tikzcd}
 & (X_1 \otimes X_2) \otimes (X_3 \otimes X_4) \arrow[dr, "a_{X_1, X_2, X_3 \otimes X_4}"] & \\
((X_1 \otimes X_2) \otimes X_3) \otimes X_4 \arrow[ur, "a_{X_1 \otimes X_2, X_3, X_4}"] \arrow[d, "a_{X_1, X_2, X_3} \otimes \mathrm{Id}"'] & & X_1 \otimes (X_2 \otimes (X_3 \otimes X_4)) \\
(X_1 \otimes (X_2 \otimes X_3)) \otimes X_4 \arrow[rr, "a_{X_1, X_2 \otimes X_3, X_4}"] & & X_1 \otimes ((X_2 \otimes X_3) \otimes X_4) \arrow[u, "\mathrm{Id} \otimes a_{X_2, X_3, X_4}"']
\end{tikzcd}$$

(Compatibility w/ $\mathbb{1}_{\mathcal{C}}$) : $\forall\ X, Y \in \mathcal{C}$: the following diagram commutes :

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{Id}"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{Id} \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

2. When $\mathcal{C} = \mathrm{Rep}(A)$: (i) having a $\otimes$ on $\mathcal{C}$ requires to have an <u>algebra hom.</u> $A \xrightarrow{\Delta} A \otimes A$ (called coproduct)

(ii) Associativity constraint - there is $\Phi \in (A \otimes A \otimes A)^{\times}$ (invertible)

(iii) Unit object - algebra hom $A \xrightarrow{\varepsilon} \mathbb{C}$ (called counit).

Requirements: • $\forall\ x \in A$, we have

$$\Phi\,(\Delta \otimes \mathrm{id})\,(\Delta(x))\,\Phi^{-1} = (\mathrm{Id} \otimes \Delta)\,(\Delta(x))$$

• $$(\varepsilon \otimes \mathrm{Id})\,(\Delta(x)) = x = (\mathrm{Id} \otimes \varepsilon)(\Delta(x))$$

• Pentagon axiom :

$$(1 \otimes 1 \otimes \Delta)(\Phi) \cdot (\Delta \otimes 1 \otimes 1)(\Phi) =$$

$$= (1 \otimes \Phi) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (\Phi \otimes 1)$$

(in $A^{\otimes 4}$)

- Compatibility of Φ with ε :

$$(1 \otimes \varepsilon \otimes 1)(\Phi) = 1 \otimes 1.$$

Given $(A, \Delta, \varepsilon, \Phi)$ - called (quasi)-bialgebra - $\mathrm{Rep}(A)$ is a tensor category as follows:

(i) $A \circlearrowright V_1, V_2$ $\rightsquigarrow$ $A \circlearrowright V_1 \otimes V_2$ by :

$\pi_j : A \longrightarrow \mathrm{End}_{\mathbb{C}}(V_j)$

$(j = 1, 2)$

$$\boxed{\begin{gathered} a \in A, \quad v_1 \in V_1, \quad v_2 \in V_2 \\ \Delta(a) = \sum_k a_k^{(1)} \otimes a_k^{(2)} \in A \otimes A \\ a \cdot (v_1 \otimes v_2) := \sum_k a_k^{(1)} \cdot v_1 \otimes a_k^{(2)} \cdot v_2 \end{gathered}}$$

i.e. $A \xrightarrow{\pi} \mathrm{End}(V_1 \otimes V_2)$ is given by

$$\pi(a) = \pi_1 \otimes \pi_2 (\Delta(a)).$$

(ii) $a_{V_1, V_2, V_3} = \pi_1 \otimes \pi_2 \otimes \pi_3 (\Phi)$ (just denoted by Φ_{V_1, V_2, V_3}).

(iii) $\mathbb{1}_{\mathrm{Rep}(A)} = \mathbb{C}$ with A-action given by

$$\varepsilon : A \longrightarrow \mathbb{C} = \mathrm{End}_{\mathbb{C}}(\mathbb{C})$$

3. MacLane's coherence theorem.

Assume $(\mathcal{C}, \otimes, a)$ is a tensor category. Let $n \in \mathbb{Z}_{\geq 2}$ and

$\mathcal{B}_n$ = set of complete bracketings on n letters.

Then for any $V_1, \dots, V_n \in \mathcal{C}$, we get a well-defined object $(V_1 \otimes \dots \otimes V_n)_b$ $\forall\, b \in \mathcal{B}_n$.

(e.g. $n=4$, $b = (\cdot\cdot)(\cdot\cdot) \rightsquigarrow (V_1 \otimes V_2) \otimes (V_3 \otimes V_4)$.)

Moreover, for $b, b' \in \mathcal{B}_n$ we have (potentially many) iso.

$$(V_1 \otimes \dots \otimes V_n)_b \longrightarrow (V_1 \otimes \dots \otimes V_n)_{b'}$$

assoc. constraint

(e.g. $n=4$, $b = ((\cdot\cdot)\cdot)\cdot$ - there are 2 ways to extend a
$b' = \cdot(\cdot(\cdot\cdot))$ - see the Pentagon axiom on page 2)

Coherence Theorem of MacLane: (assuming Pentagon axiom holds)

there is a unique isomorphism obtained from a:

$$a_{b',b} : (V_1 \otimes \dots \otimes V_n)_b \longrightarrow (V_1 \otimes \dots \otimes V_n)_{b'} \quad \forall\, b, b' \in \mathcal{B}_n.$$

Idea: $\mathcal{A}_n$: vertices are elements of $\mathcal{B}_n$
: edges are associativity constraints.
: 2-cells - faces - pentagons.

$\mathcal{A}_n$ is connected $\Rightarrow$ existence of the isomorphism.

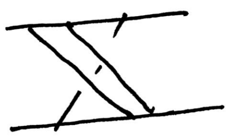
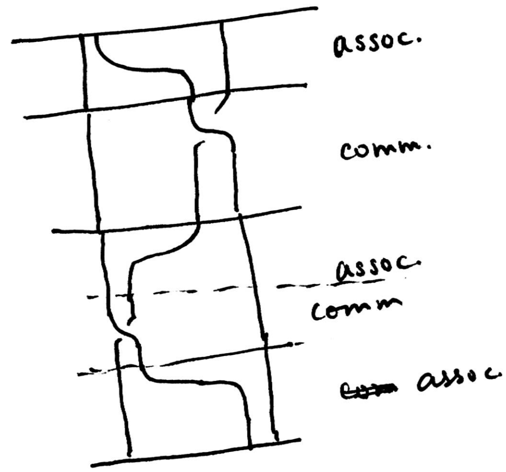
$\mathcal{A}_n$ is simply-connected $\Rightarrow$ uniqueness.

4. Braided tensor categories. Again let $(\mathcal{C}, \otimes, a)$ be a tensor category. A <u>commutativity constraint</u> c is an iso.

$$c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X \quad \forall\, X, Y \in \mathcal{C}$$

natural in X, Y. (i.e. $c : - \otimes - \longrightarrow \otimes \circ \text{flip} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.)
natural trans. of functors

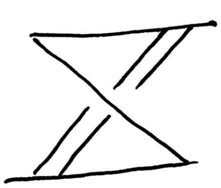
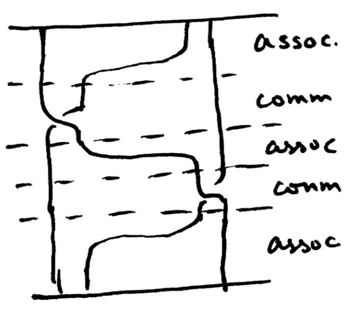
satisfying hexagon axioms:

(H1) Pictorially:  = 

i.e. $\forall\, X, Y, Z \in \mathcal{C}$:

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "c_{X \otimes Y, Z}"] \arrow[d, "a_{X,Y,Z}"'] & Z \otimes (X \otimes Y) \\
X \otimes (Y \otimes Z) \arrow[d, "\mathrm{Id} \otimes c_{Y,Z}"'] & (Z \otimes X) \otimes Y \arrow[u, "a_{Z,X,Y}"'] \\
X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[u, "c_{X,Z} \otimes \mathrm{Id}"']
\end{tikzcd}$$

commutes

(H2)  = 

(Ex. - Draw the comm. diagram for $X, Y, Z \in \mathcal{C}$ for this axiom.)

$(c \text{ vs } \mathbb{1}_{\mathcal{C}})$:

$$\begin{tikzcd} X \otimes \mathbb{1} \arrow[rr, "c_{X,\mathbb{1}}"] \arrow[dr, "r_X"'] & & \mathbb{1} \otimes X \arrow[dl, "l_X"] \\ & X & \end{tikzcd}$$

Commutes.

Braided tensor category = tensor category + commutativity constraint.

5. Let $(\mathcal{C}, \otimes, a, c)$ be a braided tensor category. Let $V \in \mathcal{C}$ and $n \in \mathbb{Z}_{\geq 2}$. For any $b \in \mathcal{B}_n \leftarrow$ bracketing on n-letters. we can define an action of the Artin's braid group B_n on $(V^{\otimes n})_b$ as follows.

Let $i \in \{1, \dots, n-1\}$. Choose $b' \in \mathcal{B}_n$ s.t. $b' = \cdots (x_i \, x_{i+1}) \cdots$

$T_i : V_b^{\otimes n} \longrightarrow V_b^{\otimes n}$ is defined as:

$$a_{bb'} \circ \left(Id^{\otimes i-1} \otimes c_{V,V} \otimes Id^{\otimes n-i-1} \right) \circ a_{b'b}.$$

That is,

$$\begin{tikzcd} V_b^{\otimes n} \arrow[r, "a_{b'b}"] \arrow[d, dashed, "T_i"'] & V_{b'}^{\otimes n} \arrow[d, "Id^{\otimes i-1} \otimes c_{V,V} \otimes Id^{\otimes n-i-1}"] \\ V_b^{\otimes n} & V_{b'}^{\otimes n} \arrow[l, "a_{bb'}"'] \end{tikzcd}$$

e.g. $n = 3$, $b = (\cdot\,\cdot)\cdot$. $\quad T_1 = c_{V,V} \otimes Id : (V \otimes V) \otimes V \to (V \otimes V) \otimes V$

$$T_2 = a_{V,V,V}^{-1} \; Id \otimes c_{VV} \; a_{V,V,V} .$$