Lecture 2

0. Recall that we started studying differential equations of the form (*): $\boxed{F'(z) = A(z)\,F(z)}$. Locally speaking, we may assume that the domain $D \subset \mathbb{C}$, is a disc around $0 \in \mathbb{C}$, and $A : D \setminus \{0\} \to M_{N\times N}(\mathbb{C})$ is a holomorphic function. Further, we assume that $A(z)$ does not have an essential singularity at 0. Thus, $A(z) = \sum_{k=-r-1}^{\infty} A_k z^k$ (Laurent series).

- $r = -1$: $A(z)$ is holomorphic near 0. In this case, we say that 0 is an <u>ordinary point</u> of (*).

  In this case (*) can be solved formally and the formal solution converges in a neighborhood of 0.

- $r = 0$: $A(z)$ has a simple pole at 0. In this case, we say that 0 is a <u>regular singular point</u> of (*).

  The fundamental solution of (*) can be written as $H(z)\cdot z^{A_{-1}}$ where $H(z) = 1 + \sum_{k=1}^{\infty} H_k z^k$ is obtained formally and then shown to be a convergent series near 0. [Assumption: eigenvalues of A_{-1} do not differ by $\mathbb{Z}_{\neq 0}$.]

- $r \geq 1$ → we say 0 is an <u>irregular singularity</u> of <u>Poincaré rank r</u>.

1. Remark. Change of variables $w = z^{-1}$, and hence $dw = -z^{-2}\,dz$, transforms $(*)$ into

$$\boxed{\frac{dF}{dw} = \frac{-1}{w^2} A\left(\frac{1}{w}\right) F} \quad (**)$$

By definition, local behaviour of $(*)$ near $z = \infty$ is the local behaviour of $(**)$ near $w = 0$.

Example : (Drinfeld ODE) $\displaystyle \frac{dF}{dz} = \left(\frac{A}{z} + \frac{B}{z-1}\right) F$

$(A, B \in M_{N\times N}(\mathbb{C}))$

$$\underset{(w = z^{-1})}{\Longrightarrow} \quad \frac{dF}{dw} = \left(\frac{-A-B}{w} + \frac{B}{w-1}\right) F.$$

Thus $z = 0$, 1 and ∞ are regular singular points of the Drinfeld ODE, with residues A, B, $-A-B$ respectively.

2. More remarks and examples of Drinfeld ODE.

$\displaystyle \frac{dF}{dz} = \left(\frac{A}{z} + \frac{B}{z-1}\right) F$ $\rightsquigarrow$ Solution near 0: $F^{(0)}(z) = H^{(0)}(z) \cdot z^{A}$

$\rightsquigarrow$ Solution near 1: $F^{(1)}(z) = H^{(1)}(z) \cdot (1-z)^{B}$

$\Rightarrow$ $\exists K$ s.t. $\boxed{F^{(0)}(z) = F^{(1)}(z) \cdot K}$

- We can consider A and B as formal parameters (non-commuting) which yields a <u>canonical series</u>

$K \in \mathbb{C}\langle\langle A, B \rangle\rangle$ (ring of formal series in 2 non-commutative variables A & B).

In this setting we note that $AB - BA = 0 \Rightarrow K(A,B) = 1$
(as $F^{(0)} = F^{(1)} = z^A (1-z)^B$).

$K(A,B)$ is, in fact, a <u>Lie series</u> (i.e., in terms of commutators $[A,B]$, $[A,[A,B]]$, etc.); with constant term 1.

- Smallest non-trivial example

$$A = \begin{bmatrix} \frac{\lambda}{2} & 0 \\ 0 & -\frac{\lambda}{2} \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$(\lambda \notin \mathbb{Z}_{\neq 0}) \leftarrow$ non-resonance condition.

$$F^{(0)}(z) = \begin{bmatrix} 1 & f(z;\lambda) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{\lambda/2} & 0 \\ 0 & z^{-\lambda/2} \end{bmatrix} \text{; where}$$

$$f(z;\lambda) = \sum_{n \geq 1} \frac{z^n}{\lambda - n} .$$

As $z \to 1$ ($z \in$ Disc around 0 of radius 1) ; $f(z;\lambda)$ does not admit a limit. A "small λ-expansion" yields

$$f(z;\lambda) = \underbrace{\ln(1-z)}_{\substack{\uparrow \\ \text{problem term} \\ \text{as } z \to 1}} \cancel{+} - \underbrace{\sum_{\ell \geq 1} \lambda^{\ell} \cdot \left(\sum_{n=1}^{\infty} \frac{z^n}{n^{\ell+1}} \right)}_{\substack{\uparrow \\ \text{Coeff of } \lambda^{\ell} \to \zeta(\ell+1) \\ \text{as } z \to 1}}$$

Thus, we get (after a few simplifications)

$$\boxed{\underset{\substack{(1,2) \\ \text{entry}}}{K} = - \sum_{\ell \geq 1} \zeta(\ell+1) \cdot \lambda^{\ell}}$$

[Here $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ Riemann ζ-function.]

3. Another non-trivial example of Drinfeld ODE.

$$A = \begin{bmatrix} \frac{\lambda}{2} & 0 \\ 0 & -\frac{\lambda}{2} \end{bmatrix} \qquad B = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}$$

Solution near $z = 0$ is of the form $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \cdot \begin{bmatrix} z^{\lambda/2} & 0 \\ 0 & z^{-\frac{\lambda}{2}} \end{bmatrix}$.

$\alpha, \beta, \gamma, \delta \in \mathbb{C}[[z]]$; $\alpha(0) = 1 = \delta(0)$ and $\beta(0) = 0 = \gamma(0)$.

To write these series explicitly, let

- $k = \sqrt{xy}$ and $r_1, r_2 = \dfrac{\lambda \pm \sqrt{\lambda^2 + 4xy}}{2}$

- $F_{2,1}(a,b;c;z) = \displaystyle\sum_{n \geq 0} \frac{z^n}{n!} \frac{(a)_n (b)_n}{(c)_n}$ where

$$(p)_n = \begin{cases} p(p+1)\cdots(p+n-1) & ; n \geq 1 \\ 1 & ; n = 0 \end{cases}$$

. Then:

Gauss Hypergeometric series.

$$\alpha = (1-z)^{-k} F_{2,1}(r_1 - k, r_2 - k; \lambda; z)$$

$$\gamma = (1-z)^{-k} F_{2,1}(r_1 - k + 1, r_2 - k + 1; \lambda + 2; z) \cdot \left(\frac{-y \cdot z}{1+\lambda} \right)$$

$$\beta = (1-z)^{k} F_{2,1}(-r_1 + k + 1, -r_2 + k + 1; 2 - \lambda; z) \cdot \left(\frac{-xz}{1-\lambda} \right)$$

$$\delta = (1-z)^{k} F_{2,1}(-r_1 + k, -r_2 + k; -\lambda; z)$$

4. Hypergeometric Series

(cf. Chapter 14 of Whittaker-Watson 'A course in modern analysis'.)

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{(a)_n (b)_n}{(c)_n} \qquad \text{(Gauss 1813)}$$

- $$\frac{d}{dz} F(a,b;c;z) = \frac{ab}{c} F(a+1,b+1;c+1;z)$$

- $$z(1-z)\frac{d^2F}{dz^2} + \left(c-(a+b+1)z\right)\frac{dF}{dz} - abF = 0$$

  (hypergeometric equation)

- $$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a,c-b;c;z)$$

- (assuming $c-a-b \notin \mathbb{Z}$) For z s.t. $\boxed{\begin{array}{l}|z|<1 \text{ and} \\ |1-z|<1\end{array}}$

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;1-(c-a-b);1-z)$$

$$+ (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(-c+a+b)}{\Gamma(a)\Gamma(b)} F(c-a,c-b;1+(c-a-b);1-z)$$

(This identity is due to Barnes (1908).)

In the last identity $\Gamma(x)$ is the Euler gamma function (cf Chapter 12 of Whittaker-Watson)

5. Gamma function. (Euler 1729)

- $\Gamma(x)$ is a meromorphic function of $x \in \mathbb{C}$ with simple poles at $x \in \mathbb{Z}_{\leq 0}$. It satisfies a difference equation

$$\Gamma(x+1) = x\,\Gamma(x)$$

- Weierstrass :

$$\frac{1}{\Gamma(x)} = x \cdot e^{\gamma x} \prod_{n=1}^{\infty} \left(1+\frac{x}{n}\right) \cdot e^{-\frac{x}{n}}$$

($\gamma = \lim\limits_{m \to \infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{m} - \ln(m)\right)$ Euler-Mascheroni constant)

- Euler :

$$\boxed{\Gamma(x)\,\Gamma(1-x) = \frac{2\pi i}{e^{\pi i x} - e^{-\pi i x}}}$$

- Euler : For $\mathrm{Re}(x) > 0$

$$\Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1}\, dt$$