0. Recall – a braided tensor category $(\mathcal{C}, \otimes, c, a)$ is the data of
   - $\mathcal{C}$: a $\mathbb{C}$-linear category
   - $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ additive bifunctor
   - a : associativity constraint satisfying pentagon axiom.
   - c : commutativity constraint satisfying hexagon axioms.

Unit object $\left(\mathbb{1}, \left(l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X\right)_{X \in \mathcal{C}}, \left(r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X\right)_{X \in \mathcal{C}}\right)$

axioms:
$$\begin{tikzcd}(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{Id}_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{Id}_X \otimes l_Y"] \\ & X \otimes Y & \end{tikzcd} \quad ; \quad \begin{tikzcd}X \otimes \mathbb{1} \arrow[rr, "c_{X,\mathbb{1}}"] \arrow[dr, "r_X"'] & & \mathbb{1} \otimes X \arrow[dl, "l_X"] \\ & X & \end{tikzcd}.$$

1. A <u>quasi-bialgebra</u> $(A, \Delta, \varepsilon, \Phi)$ is the data of
   - an assoc., unital algebra $/\mathbb{C}$
   - $\Delta : A \longrightarrow A \otimes A$ algebra hom. – called coproduct.
   - $\varepsilon : A \longrightarrow \mathbb{C}$ alg. hom. – called counit.
   - $\Phi \in A^{\otimes 3}$ invertible – called associator

s.t. (i) $\forall a \in A$, $(1 \otimes \Delta)(\Delta(a)) = \Phi \, (\Delta \otimes 1)(\Delta(a)) \, \Phi^{-1}$

(ii) Pentagon relation:
$$(1 \otimes 1 \otimes \Delta)(\Phi) \cdot (\Delta \otimes 1 \otimes 1)(\Phi)$$
$$= (1 \otimes \Phi) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (\Phi \otimes 1)$$

(iii) $\forall a \in A$, $\varepsilon \otimes 1 \, (\Delta(a)) = a = 1 \otimes \varepsilon \, (\Delta(a))$.

(iv) $(1 \otimes \varepsilon \otimes 1)(\Phi) = 1 \otimes 1$.

$(A, \Delta, \varepsilon, \Phi)$ quasi-bialgebra $\rightsquigarrow$ $\mathrm{Rep}_{(f.d.)}(A) = \mathcal{C}$ is a tensor category. In order for it to be a braided tensor category we need an invertible element $R \in A \otimes A$.

2. A <u>quasi-triangular</u> <u>quasi-bialgebra</u> $(A, \Delta, \varepsilon, R, \Phi)$ is the data of a quasi-bialgebra $(A, \Delta, \varepsilon, \Phi)$ as before, and $R \in A \otimes A$ invertible so that the following axioms hold:

[Hint: $c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$ is supposed to be $(12) \circ (\pi_V \otimes \pi_W (R))$; $\pi_V : A \to \mathrm{End}\, V \ldots$]

(Intertwining equation) $\forall\, a \in A$; $\Delta^{op}(a) = R \cdot \Delta(a) \cdot R^{-1}$.

(Hexagon relation 1)

$$\Delta \otimes 1 (R) = \cancel{\Phi_{321}}$$
$$= \Phi_{312} \cdot R_{13} \cdot \Phi^{-1}_{132} \cdot R_{23} \cdot \Phi_{123}$$

(Ex: Derive this from the hexagon axiom for (c, a) - when $c_{V,W} = (12) \circ R_{V,W}$ $a = \Phi$.)

2) $$1 \otimes \Delta (R) = \Phi^{-1}_{231}\, R_{13}\, \Phi_{213}\, R_{12}\, \Phi^{-1}_{123}$$

(Notation: $\Phi = \sum_k a_k \otimes b_k \otimes c_k$ $\rightsquigarrow$ $\Phi_{312} = \sum_k b_k \otimes c_k \otimes a_k$
$= \Phi_{123}$ and so on

$R = \sum_k x_k \otimes y_k \rightsquigarrow R_{13} = \sum_k x_k \otimes 1 \otimes y_k$)

(Counit) $\quad \varepsilon \otimes 1\,(R) = 1 \otimes 1 = 1 \otimes \varepsilon\,(R)$.

Exercise / Remark. $\quad \underset{\text{q-t-qb}}{(A, \Delta, \varepsilon, R, \Phi)} \rightsquigarrow \mathrm{Rep}(A) = \mathcal{C}$ is a

braided tensor category, with $\quad c_{X,Y} = (12) \circ \underset{\substack{\downarrow \\ \pi_X \otimes \pi_Y (R)}}{\boxed{R_{X,Y}}}$.

3. A qtqb with $\Phi = 1^{\otimes 3}$ is called <u>quasi-triangular bialgebra</u>.

The axioms then become

(Coassociativity of Δ): $\quad \Delta \otimes 1\,(\Delta(a)) = 1 \otimes \Delta\,(\Delta(a)) \quad \forall a \in A$.

Hexagon rel^{n}s - $\quad$ (1) $\Delta \otimes 1\,(R) = R_{13} R_{23}$

(2) $1 \otimes \Delta\,(R) = R_{13} R_{12}$

Interlwining equation: $\quad \Delta^{op}(a) = R\, \Delta(a)\, R^{-1} \quad \forall a \in A$.

In this case $\mathrm{Rep}(A)$ has trivial associativity constraint - sometimes refered to as - strict braided tensor category.

4. Tensor functors and twisting.

Let $(\mathcal{C}, \otimes_{\mathcal{C}}, a^{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, a^{\mathcal{D}})$ be two tensor categories, and let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor

A tensor structure on F is a natural iso.

$$J_{X,Y} : F(X \underset{\mathcal{C}}{\otimes} Y) \xrightarrow{\sim} F(X) \underset{\mathcal{D}}{\otimes} F(Y)$$, natural in X & Y.

(Thus J is a natural transformation

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow J"] & \\ \mathcal{C}\times\mathcal{C} \arrow[ur, "\otimes"] \arrow[r, "F\times F"'] & \mathcal{D}\times\mathcal{D} \arrow[r, "\otimes"'] & \mathcal{D} \end{tikzcd}$$

.). J is required to satisfy: (twist eq^n / cocycle id.)

$$\begin{tikzcd} F((X \underset{\mathcal{C}}{\otimes} Y) \underset{\mathcal{C}}{\otimes} Z) \arrow[r, "F(a^{\mathcal{C}}_{X,Y,Z})"] \arrow[d, "J_{X \underset{\mathcal{C}}{\otimes} Y, Z}"] & F(X \underset{\mathcal{C}}{\otimes} (Y \underset{\mathcal{C}}{\otimes} Z)) \arrow[d, "J_{X, Y \underset{\mathcal{C}}{\otimes} Z}"] \\ F(X \underset{\mathcal{C}}{\otimes} Y) \underset{\mathcal{D}}{\otimes} F(Z) \arrow[d, "J_{X,Y} \otimes \mathrm{id}"] & F(X) \underset{\mathcal{D}}{\otimes} F(Y \underset{\mathcal{C}}{\otimes} Z) \arrow[d, "\mathrm{id} \otimes J_{Y,Z}"] \\ (F(X) \underset{\mathcal{D}}{\otimes} F(Y)) \underset{\mathcal{D}}{\otimes} F(Z) \arrow[r, "a^{\mathcal{D}}_{F(X),F(Y),F(Z)}"'] & F(X) \underset{\mathcal{D}}{\otimes} (F(Y) \underset{\mathcal{D}}{\otimes} F(Z)) \end{tikzcd}$$

If, moreover, $\mathcal{C}$ and $\mathcal{D}$ are braided - we say J is <u>compatible with respective braidings</u>, or simply <u>J is braided</u> if it satisfies:

$$\begin{tikzcd} F(X\otimes Y) \arrow[r, "J_{X,Y}"] \arrow[d, "F(c_{X,Y})"'] & F(X)\otimes F(Y) \arrow[d, "c_{F(X),F(Y)}"] \\ F(Y\otimes X) \arrow[r, "J_{Y,X}"'] & F(Y)\otimes F(X) \end{tikzcd}$$

commutes $\forall\ X, Y \in \mathcal{C}$

5. On the algebraic side.- Let $(A, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra

Given an element $J \in A\otimes A$ (invertible), we can define

$\Delta_J : A \longrightarrow A\otimes A$ - alg. hom.

$\Delta_J(a) = J\,\Delta(a)\,J^{-1}$

$\varepsilon_J = \varepsilon$ - assuming $\boxed{\varepsilon\otimes 1(J) = 1 = 1\otimes\varepsilon(J)}$

Φ_J is defined so that • $J_{V,W} : V\otimes W \longrightarrow V\underset{J}{\otimes} W$

A-acts via Δ ; A acts via $\Delta_J = J\cdot\Delta(\,)\cdot J^{-1}$

is a tensor str. on $Id : (\mathrm{Rep}\,A, \otimes) \to (\mathrm{Rep}\,A, \underset{J}{\otimes})$

- i.e. the diagram on the previous page commutes:

$$\Phi_J = (1\otimes J)\cdot 1\otimes\Delta(J)\cdot\Phi\cdot\Delta\otimes 1(J)^{-1}\cdot(J^{-1}\otimes 1)$$

$(A, \Delta, \varepsilon, \Phi) \rightsquigarrow \boxed{\begin{array}{l} J\in A\otimes A \\ \text{invertible} \\ \text{s.t.} \\ \varepsilon\otimes 1(J) = 1 \\ \quad = 1\otimes\varepsilon(J)\end{array}} \rightsquigarrow (A, \Delta_J, \varepsilon, \Phi_J)$

called "twist".

twist equivalent to $(A, \Delta, \varepsilon, \Phi)$

If $R \in (A \otimes A)^\times$ makes $(A, \Delta, \varepsilon, \Phi, R)$ into a q-t-q-b,

then $R_J := J_{21}\, R\, J^{-1}$ is "the R-matrix" for $(A, \Delta_J, \varepsilon, \Phi_J)$

(easy check: $\Delta_J^{op}(a) = J_{21}\ \Delta^{op}(a)\ J_{21}^{-1}$

$$= J_{21}\ R\, \Delta(a)\, R^{-1}\, J_{21}^{-1}$$

$$= J_{21}\, R\, J^{-1} \cdot (J\, \Delta(a)\, J^{-1})\, J\, R^{-1}\, J_{21}^{-1}$$

$$= R_J \cdot \Delta_J(a) \cdot R_J^{-1} \ . \quad)$$

6. "Trivial" example. - $\mathcal{C} = Rep(\mathfrak{g})$ when $\mathfrak{g}$ is a Lie algebra

$\otimes$ on $\mathcal{C}$: $\mathfrak{g} \circlearrowright V_1, V_2 \Rightarrow \mathfrak{g} \circlearrowright V_1 \otimes V_2$ by

$$x \cdot (v_1 \otimes v_2) = (x \cdot v_1) \otimes v_2 + v_1 \otimes (x \cdot v_2)$$

$\mathbb{1}_{\mathcal{C}} = \mathbb{C} \circlearrowleft \mathfrak{g}$ by $x \cdot \equiv 0$.

assoc. constraint $=$ natural id. of vector spaces

$$(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$$

comm. constraint $=$ flip: $V \otimes W \to W \otimes V$

$$v \otimes w \mapsto w \otimes v$$

Algebra $A = \mathcal{U}(\mathfrak{g}) =$ universal enveloping algebra of $\mathfrak{g}$

$$= \frac{\text{Tensor algebra of } \mathfrak{g} = \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathfrak{g}^{\otimes n} \text{ w/ mult.} = \text{concatenation of tensors}}{\text{Two sided ideal generated by } x \otimes y - y \otimes x - [x, y] \ : \ x, y \in \mathfrak{g}.}$$

(e.g. $\mathfrak{g} = \mathfrak{sl}_2 \rightsquigarrow \mathcal{U}(\mathfrak{g}) =$ unital assoc. alg / $\mathbb{C}$ generated by

e, f, h ; subject to rel^{n}s

$$he - eh = 2e$$
$$hf - fh = -2f$$
$$ef - fe = h$$

).

$\Delta : \mathcal{U}(\mathfrak{a}) \longrightarrow \mathcal{U}(\mathfrak{a}) \otimes \mathcal{U}(\mathfrak{a})$ defined by ; $\varepsilon : \mathcal{U}(\mathfrak{a}) \to \mathbb{C}$

$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad \forall\, x \in \mathfrak{a}.$$

$\varepsilon(x) = 0 \ \forall\, x \in \mathfrak{a}$.

Thus $\left(\mathcal{U}(\mathfrak{a}), \Delta, \varepsilon, R = 1^{\otimes 2}, \Phi = 1^{\otimes 3}\right)$ – an example

co-commutative bialgebra.