Lecture 21

0. Let $\mathfrak{g}$ be a (finite dimensional) Lie algebra / $\mathbb{C}$ and $(\cdot,\cdot):\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ a symmetric, bilinear, non-degenerate, invariant form.

Let $\Omega \in \mathfrak{g}\otimes\mathfrak{g}$ be its canonical tensor.

Recall: (i) $[x\otimes 1 + 1\otimes x, \Omega] = 0 \quad \forall\, x\in\mathfrak{g}$. (Lecture 18, Lemma 1)

(ii) Let $\kappa\in\mathbb{C}$ be a parameter and $n\in\mathbb{Z}_{\geq 2}$.

$$\nabla_{KZ_n} = d - \kappa \sum_{1\leq i<j\leq n} \frac{d(z_i - z_j)}{z_i - z_j}\,\Omega_{ij} \qquad \text{(Lecture 18, §2.)}$$

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Thus,

$$\mathcal{U}(\mathfrak{g}) = \frac{\text{free unital associative algebra generated by } \mathfrak{g}}{\text{Two-sided ideal gen. by } x\cdot y - y\cdot x - [x,y] \ \ \forall\, x,y\in\mathfrak{g}.}$$

$$\left.\begin{array}{ll}\text{Coproduct}: & \Delta_0(x) = x\otimes 1 + 1\otimes x \quad \forall\, x\in\mathfrak{g}.\\ \text{Counit}: & \varepsilon_0(x) = 0 \quad \forall\, x\in\mathfrak{g}.\end{array}\right\}\ \text{algebra hom-s.}$$

To make sense of R_{KZ} and Φ_{KZ} in the algebraic setting, we treat κ from (ii) above as a formal parameter - often rescaled as $\kappa = \dfrac{\hbar}{2\pi i}$. Then $R_{KZ} = e^{\pi i\kappa\Omega} = e^{\frac{\hbar}{2}\Omega} \in (\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g}))[[\hbar]]$

$\Phi_{KZ} \in \mathcal{U}(\mathfrak{g})^{\otimes 3}[\![\hbar]\!]$ is then the associator for

$$\frac{dF}{dz} = \frac{\hbar}{2\pi i}\left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z-1}\right)F$$

I will often write $\Phi_{KZ} = \Phi(\Omega_{12}, \Omega_{23})$. Now we will prove that:

Theorem.- $(\mathcal{U}(\mathfrak{g})[\![\hbar]\!], \underset{0}{\Delta}, \varepsilon_0, R_{KZ}, \Phi_{KZ})$ is a quasi-triangular quasi-bialgebra.

The same proof gives a structure of braided tensor category on $Rep_{fd}(\mathfrak{g})$ – assuming $k \in \mathbb{C}$ is "generic" – so that the resonance condition for the existence/uniqueness of the fundamental solution holds
[Note - eigenvalues of Ω on $V_1 \otimes V_2$ are integral – so it is enough to consider $k \notin \mathbb{Q}$.]

1. By definition we need to prove the following:

(a) $\varepsilon \otimes 1(\Delta(x)) = x = 1 \otimes \varepsilon(x) \quad \forall x \in \mathfrak{g}$.

[True by direct inspection]

(b) $$\left.\begin{aligned} \Delta^{op}(x) &= R\,\Delta(x)\,R^{-1} \\ 1 \otimes \Delta(\Delta(x)) &= \Phi\ \Delta \otimes 1(\Delta(x))\ \Phi^{-1} \end{aligned}\right\} \forall x \in \mathfrak{g}.$$

Proof. Since $\Delta(x) = \Delta^{op}(x) = x \otimes 1 + 1 \otimes x$ commutes w/ Ω and $R_{KZ} = e^{\frac{\hbar}{2}\Omega}$; $\Delta(x)$ commutes w/ R_{KZ}

$$\Rightarrow \quad \Delta^{op}(x) = R_{KZ}\, \Delta(x)\, R_{KZ}^{-1}.$$

Similarly $\quad 1\otimes\Delta(\Delta(x)) = \Delta\otimes 1(\Delta(x))$

$$= x\otimes 1\otimes 1 + 1\otimes x\otimes 1 + 1\otimes 1\otimes x$$

commutes with both Ω_{12} and Ω_{23} and hence with Φ_{KZ}. $\square$

(c) $\quad \varepsilon\otimes 1\,(R) = 1 = 1\otimes\varepsilon\,(R)$

$\quad 1\otimes\varepsilon\otimes 1\,(\Phi) = 1\otimes 1.$

[This is true since $\quad \varepsilon\otimes 1\,(\Omega) = 0 = 1\otimes\varepsilon(\Omega)$

and $\quad \Phi(A,B)\big|_{A=B=0} = 1.$]

(d) Pentagon and hexagon axioms. – below.

2. Pentagon axiom.

$$(1\otimes 1\otimes\Delta)(\Phi)\cdot(\Delta\otimes 1\otimes 1)(\Phi) = (1\otimes\Phi)\cdot(1\otimes\Delta\otimes 1)(\Phi)\cdot(\Phi\otimes 1).$$

$$\begin{tikzcd}
 & (V_1\otimes V_2)\otimes(V_3\otimes V_4) \arrow[dr, "1\otimes 1\otimes\Delta(\Phi)"] & \\
((V_1\otimes V_2)\otimes V_3)\otimes V_4 \arrow[ur, "\Delta\otimes 1\otimes 1(\Phi)"] \arrow[d, "\Phi\otimes 1"'] & & V_1\otimes(V_2\otimes(V_3\otimes V_4)) \\
(V_1\otimes(V_2\otimes V_3))\otimes V_4 \arrow[rr, "1\otimes\Delta\otimes 1(\Phi)"'] & & V_1\otimes(V_2\otimes V_3)\otimes V_4 \arrow[u, "1\otimes\Phi"']
\end{tikzcd}$$

Idea: for each bracketing b on 4 letters – there exists a fundamental soln. of $\nabla_{KZ_4} \cdot \psi = 0$, denoted by ψ_b, so that $\phi_{b',b} = \psi_{b'}^{-1} \psi_b$.

3. Example $b_1 = ((\cdot\cdot)\cdot)\cdot$ Asymptotic Zone $z_2 - z_1 \ll z_3 - z_1 \ll z_4 - z_1$

Change of variables:

$$u = z_4 - z_1 \qquad\qquad z_4 - z_1 = u$$

$$v = \frac{z_3 - z_1}{z_4 - z_1} \qquad\qquad z_3 - z_1 = u.v$$

$$w = \frac{z_2 - z_1}{z_3 - z_1} \qquad\qquad z_2 - z_1 = u v w$$

$$\Rightarrow \quad z_3 - z_2 = uv(1-w)$$

$$z_4 - z_2 = u(1 - vw)$$

$$z_4 - z_3 = u(1-v)$$

and ∇_{KZ_4} becomes

$$\nabla_{KZ_4} = d - \kappa \left(\frac{du}{u} \Omega_{[14]} + \frac{dv}{v} \Omega_{[13]} + \frac{dw}{w} \Omega_{12} + \text{terms regular at } v = w = 0 \right)$$

$\Rightarrow$ Solution of the form

$$\psi_1 = H(v, w) \cdot u^{\kappa \Omega_{[14]}} \, v^{\kappa \Omega_{[13]}} \, w^{\kappa \Omega_{12}}$$

(Note: here and below: $\Omega_{[ab]} = \sum_{a \leq i < j \leq b} \Omega_{ij}$.)

4. Carrying out the computation as in §3 above we find 5 solutions of $\nabla_{KZ_4} \Psi = 0$:

Bracketing b	Asymptotic Zone Σ_b	Multivalued term of Ψ_b
1. $((\cdot\cdot)\cdot)\cdot$	$z_2 - z_1 \ll z_3 - z_1 \ll z_4 - z_1$	$\left(\frac{z_2 - z_1}{z_3 - z_1}\right)^{k\Omega_{12}} \left(\frac{z_3 - z_1}{z_4 - z_1}\right)^{k\Omega_{[13]}} (z_4 - z_1)^{k\Omega_{[14]}}$
2. $(\cdot(\cdot\cdot))\cdot$	$z_3 - z_2 \ll z_3 - z_1 \ll z_4 - z_1$	$\left(\frac{z_{32}}{z_{31}}\right)^{k\Omega_{23}} \left(\frac{z_{31}}{z_{41}}\right)^{k\Omega_{[13]}} (z_{41})^{k\Omega_{[14]}}$
3. $\cdot((\cdot\cdot)\cdot)$	$z_3 - z_2 \ll z_4 - z_2 \ll z_4 - z_1$	$\left(\frac{z_{32}}{z_{42}}\right)^{k\Omega_{23}} \left(\frac{z_{42}}{z_{41}}\right)^{k\Omega_{[24]}} (z_{41})^{k\Omega_{[14]}}$
4. $\cdot(\cdot(\cdot\cdot))$	$z_4 - z_3 \ll z_4 - z_2 \ll z_4 - z_1$	$\left(\frac{z_{43}}{z_{42}}\right)^{k\Omega_{34}} \left(\frac{z_{42}}{z_{41}}\right)^{k\Omega_{[24]}} (z_{41})^{k\Omega_{[14]}}$
5. $(\cdot\cdot)(\cdot\cdot)$	$z_2 - z_1 \ll z_4 - z_1$ $z_4 - z_3 \ll z_4 - z_1$	$\left(\frac{z_{21}}{z_{41}}\right)^{k\Omega_{12}} \left(\frac{z_{43}}{z_{41}}\right)^{k\Omega_{34}} (z_{41})^{k\Omega_{[14]}}$

$(z_{ba} = z_b - z_a)$

We claim that
$$\Psi_1 = \Psi_2 \cdot (\Phi \otimes 1)$$
$$\Psi_3 = \Psi_4 \cdot (1 \otimes \Phi) \quad \text{and}$$

$$\Psi_2 = \Psi_3 \cdot \left(1 \otimes \Delta \otimes 1\, (\Phi)\right) ; \quad \Psi_1 = \Psi_5 \left(\Delta \otimes 1 \otimes 1\, (\Phi)\right) ; \quad \Psi_5 = \Psi_4 \left(1 \otimes 1 \otimes \Delta (\Phi)\right)$$

5. Proof of $\Psi_1 = \Psi_2 \cdot (\Phi \otimes 1)$:

From the table above : $\Psi_1 \sim (z_2 - z_1)^{k\Omega_{12}} (z_3 - z_1)^{k(\Omega_{13}+\Omega_{23})} (z_4 - z_1)^{k(\Omega_{14}+\Omega_{24}+\Omega_{34})}$

$$\Psi_2 \sim (z_3 - z_2)^{k\Omega_{23}} (z_3 - z_1)^{k(\Omega_{12}+\Omega_{13})} (z_4 - z_1)^{k(\Omega_{14}+\Omega_{24}+\Omega_{34})}$$

Let
$$W_1 = \Psi_1 \cdot (z_4 - z_1)^{-k(\Omega_{14}+\Omega_{24}+\Omega_{34})}$$

$$W_2 = \Psi_2 \, (\Phi \otimes 1)\, (z_4 - z_1)^{-k(\Omega_{14}+\Omega_{24}+\Omega_{34})}$$
$$= \Psi_2 \, (z_4 - z_1)^{-k(\Omega_{14}+\Omega_{24}+\Omega_{34})} (\Phi \otimes 1)$$

(since $\Phi \otimes 1$ only depends on Ω_{12} and Ω_{23} and these commute with $\Omega_{14} + \Omega_{24} + \Omega_{34}$.)

Thus both W_1 and W_2 satisfy :

$$\frac{\partial W}{\partial z_1} = k \sum_{j \neq 1} \frac{\Omega_{1j}}{z_1 - z_j} W + k \cdot W \cdot \frac{\Omega_{14}+\Omega_{24}+\Omega_{34}}{z_4 - z_1}$$

$$\frac{\partial W}{\partial z_i} = k \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} W$$
$(i = 2, 3)$

$$\frac{\partial W}{\partial z_4} = k \sum_{j \neq 4} \frac{\Omega_{4j}}{z_4 - z_j} W - k \cdot W \cdot \frac{\Omega_{14}+\Omega_{24}+\Omega_{34}}{z_4 - z_1} \quad - (*)$$

and both W_1, W_2 admit a limit as $z_4 \to \infty$.

Let $z_4 \to \infty$ in the system of diff. eq^{n}s above to see that

$$\left.\begin{array}{l} W_1(z_1,z_2,z_3,\infty) \\ W_2(z_1,z_2,z_3,\infty) \end{array}\right\} \text{ solve } KZ_3 \qquad \begin{array}{l} W_1 \leftrightarrow (\cdot\cdot)\cdot \\ W_2 \leftrightarrow [\cdot(\cdot\cdot)]\cdot \Phi\otimes 1. \end{array}$$

$$\Rightarrow \quad W_1(z_1,z_2,z_3,\infty) = W_2(z_1,z_2,z_3,\infty)$$

This, and the diff'l eq^n $(*)$ in z_4 variable $\Rightarrow$ $W_1 = W_2$ everywhere. $\square$

The proof of $\Psi_3 = \Psi_4 \cdot (1\otimes\Phi)$ is similar, hence omitted.

6. Proof of $\Psi_2 = \Psi_3 \cdot (1\otimes\Delta\otimes 1)(\Phi)$.

Again, recall: $\Psi_2 \sim (z_3-z_2)^{k\Omega_{23}} (z_3-z_1)^{k(\Omega_{12}+\Omega_{13})} (z_4-z_1)^{k(\Omega_{14}+\Omega_{24}+\Omega_{34})}$

$\Psi_3 \sim (z_3-z_2)^{k\Omega_{23}} (z_4-z_2)^{k(\Omega_{24}+\Omega_{34})} (z_4-z_1)^{k(\Omega_{14}+\Omega_{24}+\Omega_{34})}$

Let $V_2 = \Psi_2 \cdot (z_3-z_2)^{-k\Omega_{23}}$

$$\begin{aligned} V_3 &= \Psi_3 \cdot (1\otimes\Delta\otimes 1)(\Phi)\cdot(z_3-z_2)^{-k\Omega_{23}} \\ &= \Psi_3 \cdot (z_3-z_2)^{-k\Omega_{23}} (1\otimes\Delta\otimes 1)(\Phi) \end{aligned}$$

$\Bigg($ Since $\left.\begin{array}{l} 1\otimes\Delta\otimes 1\,(\Omega_{12}) = \Omega_{12}+\Omega_{13} \\ 1\otimes\Delta\otimes 1\,(\Omega_{23}) = \Omega_{24}+\Omega_{34} \end{array}\right\}$ commute w/ Ω_{23}

$\Rightarrow$ $1\otimes\Delta\otimes 1\,(\Phi)$ commutes with Ω_{23} $\Bigg)$.

- V_2 and V_3 admit $z_2 \to z_3$ limit, and solve:

$$\frac{\partial V}{\partial z_i} = k \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} V \qquad (i = 1 \text{ or } 4)$$

$$\frac{\partial V}{\partial z_2} = k \sum_{j \neq 2,3} \frac{\Omega_{2j}}{z_2 - z_j} V + k \frac{[\Omega_{23}, V]}{z_2 - z_3},$$

$$\frac{\partial V}{\partial z_3} = k \sum_{j \neq 2,3} \frac{\Omega_{3j}}{z_3 - z_j} V - k \frac{[\Omega_{23}, V]}{z_2 - z_3}.$$

Set $T_j(z_1, z_2, z_4) = V_j(z_1, z_2, z_2, z_4)$. Then we get
$(j = 2, 3)$

$$\left.\begin{aligned}
\frac{\partial T}{\partial z_1} &= k \left(\frac{\Omega_{12} + \Omega_{13}}{z_1 - z_2} + \frac{\Omega_{14}}{z_1 - z_4} \right) T \\
\frac{\partial T}{\partial z_2} &= k \left(\frac{\Omega_{12} + \Omega_{13}}{z_2 - z_1} + \frac{\Omega_{24} + \Omega_{34}}{z_2 - z_4} \right) T \\
\frac{\partial T}{\partial z_4} &= k \left(\frac{\Omega_{14}}{z_4 - z_1} + \frac{\Omega_{24} + \Omega_{34}}{z_4 - z_2} \right) T
\end{aligned}\right\}$$

Same as KZ_3 with $1 \otimes \Delta \otimes 1$ applied to the coefficients.

Hence $(V_2 = V_3)\Big|_{z_2 = z_3}$

Combined with the differential eq$^{\underline{n}}$s above, we get $V_2 = V_3$ everywhere

(proofs of $\Psi_1 = \Psi_5(\Delta \otimes 1 \otimes 1(\Phi))$
$\Psi_5 = \Psi_4(1 \otimes 1 \otimes \Delta(\Phi))$ are similar.) $\square$