Lecture 22

0. Recall: we started with $(\mathfrak{g}, (\cdot,\cdot))$ = f.d. Lie algebra w/ non-degenerate symmetric bilinear form; $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ Casimir tensor

$\Big($ e.g. $\mathfrak{g} = sl_2$; $\begin{matrix}(h,h) = 2 \\ (e,f) = 1\end{matrix}$ all other 0; $\Omega = \frac{h \otimes h}{2} + e \otimes f + f \otimes e \Big)$.

Properties of Ω:
- $[x \otimes 1 + 1 \otimes x, \Omega] = 0 \quad \forall x \in \mathfrak{g}$
- $\Omega_{21} = \Omega \quad$ (symmetric)
- $\Delta \otimes 1 (\Omega) = \Omega_{13} + \Omega_{23}$.

  $1 \otimes \Delta (\Omega) = \Omega_{12} + \Omega_{13}$.

Given k a parameter, we defined $R_{KZ} = e^{\pi i k \Omega}$ $\Big(= e^{\frac{\hbar}{2}\Omega} \in \mathcal{U}(\mathfrak{g})^{\otimes 2}[[\hbar]]$ or $GL(V_1 \otimes V_2)\Big)$

$[k = \hbar / 2\pi i]$

$$\Phi_{KZ}(k\Omega_{12}, k\Omega_{23}) = \text{associator of } F'(z) = \left(\frac{k\Omega_{12}}{\cancel{z_1 - z_2}\; z} + \frac{k\Omega_{23}}{z-1}\right)F$$

1. Recall $\displaystyle \nabla_{KZ_n} = d - k \sum_{1 \leq i < j \leq n} \frac{d(z_i - z_j)}{z_i - z_j}\Omega_{ij}$.

$(n \in \mathbb{Z}_{\geq 2})$

Now for each $b \in \mathcal{B}_n$ (set of complete bracketings on n letters) we have a multi-valued function $\mathcal{E}_b(z_1, \dots, z_n)$ – for $n = 3$, e.g.:

$$\mathcal{E}_{(\cdot\cdot)\cdot}(z_1, z_2, z_3) = \left(\frac{z_2 - z_1}{z_3 - z_1}\right)^{k\Omega_{12}} (z_3 - z_1)^{k(\Omega_{12} + \Omega_{23} + \Omega_{13})}$$

$$\mathcal{E}_{\cdot(\cdot\cdot)}(z_1, z_2, z_3) = \left(\frac{z_3 - z_2}{z_3 - z_1}\right)^{k\Omega_{23}} (z_3 - z_1)^{k(\Omega_{12} + \Omega_{23} + \Omega_{13})}$$

Since we only need the following result for $n=4$, $\mathcal{E}_b(\underline{z})$ are given in the table on page 5, Lecture 21; and it was proved in the last lecture:

Theorem. $\forall\ b \in \mathcal{B}_4$, $\exists!\ \psi_b(z_1,z_2,z_3,z_4)$ - $\mathcal{U}(\mathfrak{g})^{\otimes 4}[\![\hbar]\!]$-valued (or $GL(V_1 \otimes V_2 \otimes V_3 \otimes V_4)$-valued) mero. function s.t.

- $\nabla_{KZ_4} \cdot \psi_b = 0$
- $\lim\limits_{\boxed{\underline{z} \to \infty \text{ as prescribed by } b}} \psi_b(\underline{z}) \cdot \mathcal{E}_b(\underline{z})^{-1} = 1^{\otimes 4}$

e.g. $\underline{b} = (\cdot\cdot)(\cdot\cdot)$; it is enough to let $\dfrac{z_2 - z_1}{z_4 - z_1}, \dfrac{z_4 - z_3}{z_4 - z_1} \to 0$.

We also proved that (see the table on page 5 lecture 21 - for labels)

$$\left.\begin{aligned} \psi_1 &= \psi_2 \cdot (\Phi \otimes 1) \\ \psi_2 &= \psi_3 \cdot (1 \otimes \Delta \otimes 1)(\Phi) \\ \psi_3 &= \psi_4 \cdot (1 \otimes \Phi) \end{aligned}\right\} \Rightarrow \psi_1 = \psi_4 \cdot (1 \otimes \Phi) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (\Phi \otimes 1)$$

Also:

$$\left.\begin{aligned} \psi_1 &= \psi_5 \left(\Delta \otimes 1 \otimes 1 (\Phi)\right) \\ \psi_5 &= \psi_4 \left(1 \otimes 1 \otimes \Delta (\Phi)\right) \end{aligned}\right\} \Rightarrow \psi_1 = \psi_4 \cdot \left(1 \otimes 1 \otimes \Delta(\Phi) \cdot \Delta \otimes 1 \otimes 1 (\Phi)\right)$$

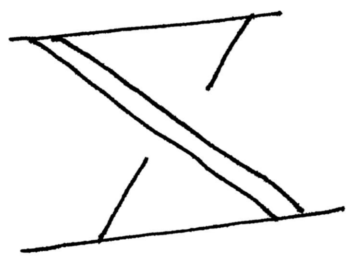
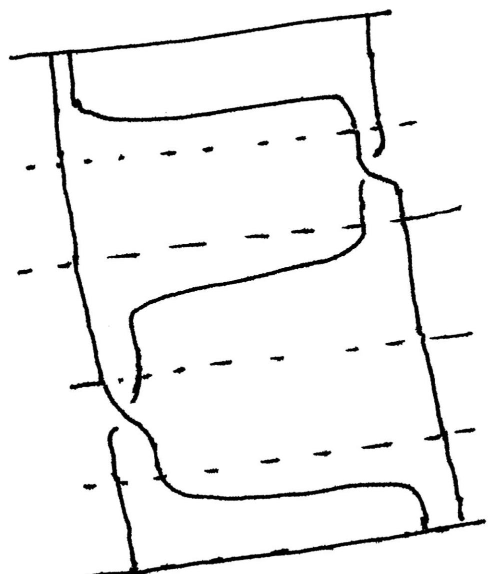
Hence we obtain the pentagon relation. (since both sides are $\psi_4^{-1} \cdot \psi_1$)

$$1 \otimes 1 \otimes \Delta(\Phi) \cdot \Delta \otimes 1 \otimes 1 (\Phi)$$
$$= (1 \otimes \Phi)\left(1 \otimes \Delta \otimes 1 (\Phi)\right)(\Phi \otimes 1).$$

2. Hexagon relations.

$$\Delta\otimes 1\,(R) = \Phi_{312}\cdot R_{13}\,,\ \Phi^{-1}_{132}\ R_{23}\ \Phi_{123}\,.$$

Picture again:

 $=$ 

The idea of the proof is the same as before. Namely, we get to compare 6 functions

$$\psi_0 \sim \left(\frac{z_2-z_1}{z_3-z_1}\right)^{k\Omega_{12}} (z_3-z_1)^{k\Omega_{[13]}} \quad \text{as } |z_2-z_1| \ll |z_3-z_1|$$

$$\psi_1 \sim \left(\frac{z_3-z_2}{z_3-z_1}\right)^{k\Omega_{23}} (z_3-z_1)^{k\Omega_{[13]}} \quad \text{as } |z_3-z_2| \ll |z_3-z_1|$$

– solutions of $\left(d - k\left(d\log(z_1-z_2)\,\Omega_{12} + d\log(z_2-z_3)\,\Omega_{23} + d\log(z_1-z_3)\,\Omega_{13}\right)\right)\psi = 0$

– definition of Φ: $\boxed{\psi_0 = \psi_1\cdot\Phi}$

$$\psi_2 \sim \left(\frac{z_2 - z_3}{z_2 - z_1}\right)^{k\Omega_{23}} (z_2 - z_1)^{k\Omega_{[13]}} \quad \text{as } |z_2 - z_3| \ll |z_2 - z_1|$$

$$\psi_3 \sim \left(\frac{z_3 - z_1}{z_2 - z_1}\right)^{k\Omega_{13}} (z_2 - z_1)^{k\Omega_{[13]}} \quad \text{as } |z_3 - z_1| \ll |z_2 - z_1|$$

- solutions of $(23)\nabla_{KZ}$:

$$d\psi = k\left(d\log(z_1 - z_2)\cdot\Omega_{13} + d\log(z_2 - z_3)\,\Omega_{23} + d\log(z_1 - z_3)\,\Omega_{12}\right)\psi$$

- $$\left.\begin{aligned} \psi_3 &= \psi_2 \cdot \Phi_{132} \\ \psi_1 &= \psi_2 \cdot R_{23} \end{aligned}\right\} \text{ by definition}$$

Finally

$$\psi_4 \sim \left(\frac{z_1 - z_3}{z_2 - z_3}\right)^{k\Omega_{13}} (z_2 - z_3)^{k\Omega_{[13]}} \quad ; \ |z_1 - z_3| \ll |z_2 - z_3|$$

$$\psi_5 \sim \left(\frac{z_2 - z_1}{z_2 - z_3}\right)^{k\Omega_{12}} (z_2 - z_3)^{k\Omega_{[13]}} \quad ; \ |z_1 - z_2| \ll |z_2 - z_3|$$

- solns of $d\psi = k\left(d\log(z_1 - z_2)\,\Omega_{13} + d\log(z_2 - z_3)\,\Omega_{12} + d\log(z_1 - z_3)\,\Omega_{23}\right)\psi$

- $$\left.\begin{aligned} \psi_4 &= \psi_5 \cdot \Phi_{312} \\ \psi_3 &= \psi_4 \cdot R_{13} \end{aligned}\right\} \text{ by defn.}$$

Combining, we get

$$\psi_0 = \psi_5 \cdot \Phi_{312} \, R_{13} \, \Phi_{132}^{-1} \, R_{23} \, \Phi_{123}$$

Compare ψ_0 and ψ_5 directly as follows: both functions admit a limit as $z_1 \to z_2$; if we remove the factor $(z_2 - z_1)^{k\Omega_{12}}$

$\Omega_{13}+\Omega_{23} \cong \Delta\otimes 1(\Omega)$

i.e. $\lim\limits_{z_1 \to z_2} \psi_0 \cdot (z_2 - z_1)^{-k\Omega_{12}}$ gives $(z_3 - z_1)^{k(\Omega_{13}+\Omega_{23})}$

$\lim\limits_{z_1 \to z_2} \psi_5 \cdot (z_2 - z_1)^{-k\Omega_{12}}$ gives $(z_1 - z_3)^{k(\Omega_{13}+\Omega_{23})}$

related by $(-1)^{k\cdot \Delta\otimes 1(\Omega)} = e^{\pi i k \cdot \Delta\otimes 1(\Omega)}$

Also, $(\Delta \otimes id)(R)$ commutes w/ Ω_{12}

$$\Rightarrow \psi_0 = \psi_5 \cdot (\Delta \otimes 1)(R)$$

Hence we obtain the 1st hexagon axiom. The second one

is proved similarly.