Lecture 23

0. Quantum group $\mathcal{U}_\hbar(\mathfrak{g})$ - $\mathfrak{g}$ is a f.d. simple Lie alg / $\mathbb{C}$ (associated to the data of a root system).

- is an example of <u>quasi-triangular Hopf algebras</u>

1. Definition. — A <u>quasi-triangular bialgebra</u> is $(A, \Delta, \varepsilon, R)$ where
   - A : unital associative algebra (will be over $\mathbb{C}[[\hbar]]$ later) over $\mathbb{C}$
   - $\Delta : A \to A \otimes A$ is an algebra hom. s.t.

(coassociativity) $\Delta \otimes \mathrm{Id}(\Delta(a)) = \mathrm{Id} \otimes \Delta(\Delta(a)) \quad \forall a \in A$

   - $\varepsilon : A \to \mathbb{C}$ alg. hom. s.t

$$\varepsilon \otimes \mathrm{id}(\Delta(a)) = a = \mathrm{id} \otimes \varepsilon(\Delta(a)) \quad \forall a \in A$$

$$(1 \otimes_{\mathbb{C}} a \in \mathbb{C} \otimes_{\mathbb{C}} A \cong A.)$$

and $R \in A \otimes A$ is an invertible element satisfying

$$\Delta^{op}(a) = R\, \Delta(a)\, R^{-1} \quad \text{- (intertwining eq}^{n}\text{)}$$

$$\Delta \otimes \mathrm{Id}(R) = R_{13} \cdot R_{23} \quad \text{(cabling identities)}$$

$$\mathrm{Id} \otimes \Delta(R) = R_{13} \cdot R_{12}$$

$$\varepsilon \otimes \mathrm{Id}(R) = 1 = \mathrm{id} \otimes \varepsilon(R).$$

2. A <u>quasi-triangular Hopf algebra</u> is $(A, \Delta, \varepsilon, S, R)$ where

$(A, \Delta, \varepsilon, R)$ = quasi-triangular bialgebra as before;

and $S : A \to A$ (called antipode) is algebra anti-automorphism (i.e. S is invertible & $S(ab) = S(b)S(a)$ $\forall a, b \in A$.)

satisfying the following axioms

$$\text{mult} \circ (S \otimes \mathrm{Id})(\Delta(x)) = \varepsilon(x). \tag{2.1}$$
$$= \text{mult} \circ (\mathrm{Id} \otimes S)(\Delta(x))$$

e.g. $A = \mathcal{U}(\mathfrak{g})$; $\Delta(x) = x \otimes 1 + 1 \otimes x \quad \forall x \in \mathfrak{g}$

$$\varepsilon(x) = 0 \quad \forall x \in \mathfrak{g}$$

has the following antipode, $S(x) = -x \quad \forall x \in \mathfrak{g}$.

Remark (1) S allows us to define <u>dual representations</u> :

$$A \circlearrowright V \rightsquigarrow A \circlearrowright V^* \qquad (a \cdot \xi)(v) = \xi(S(a)v)$$

$$\text{or } A \circlearrowright {}^*V \qquad (a \cdot \xi)(v) = \xi(S^{-1}(a) \cdot v)$$

[left / right duals.]

In $\mathcal{U}(\mathfrak{g})$ - example, both these actions coincide, but in general this need not be the case.

Remark (2). It follows from the axioms that :

Lemma. $S \otimes 1 (R) = R^{-1}$

Proof. Use $\varepsilon \otimes 1 (R) = 1$ and $\mu((S \otimes 1)\Delta(x)) = \varepsilon(x)$

$\Delta \otimes 1 (R) = R_{13} R_{23}$ (μ = mult. $a \otimes b \mapsto a \cdot b$)

Apply $S \otimes 1 \otimes 1$ to $\Delta \otimes 1 (R) = R_{13} R_{23}$ and multiply first two factors to get $\varepsilon \otimes 1 (R) = S \otimes 1 (R) \cdot R$.

Hence the lemma. $\square$

3. Thus quasi-triangular bialgebra A- has trivial associator.

- i.e. $\mathcal{C} = \operatorname{Rep} A$, $\otimes$ from $\Delta : A \to A \otimes A$,

$c_{V,W} = (12) \circ R_{V,W}$: comm. constraint,

a_{V_1, V_2, V_3} = natural id. of vector spaces

$(V_1 \otimes V_2) \otimes V_3 \simeq V_1 \otimes (V_2 \otimes V_3)$

- is a braided $\otimes$ category

(strict $\leftrightarrow$ trivial assoc constraint).

Prop.- R satisfies <u>Yang-Baxter equation</u> - the following identity in $A \otimes A \otimes A$.

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

Proof
$$R_{12} R_{13} R_{23} = R_{12} \cdot \Delta \otimes id (R)$$
$$= \Delta^{op} \otimes id (R) \cdot R_{12} = R_{23} R_{13} R_{12}. \square$$

Cor. $(A, \Delta, \varepsilon, R)$ - quasi-triangular bialgebra;

$$A \circlearrowright V \Rightarrow B_n \circlearrowright V^{\otimes n} \quad \forall\, n \in \mathbb{Z}_{\geq 2}.$$

$$\Big(T_i \longmapsto (i\ i{+}1)\, \underset{\substack{\uparrow \\ \text{acts only on } i^{th} \,\&\, (i+1)^{st} \otimes\text{-component.}}}{R_{i,i+1}} \cdot \Big)$$

Pf. - To check $B_n \longrightarrow GL(V^{\otimes n})$ is a group hom.

$$T_i \longmapsto (i\ i{+}1)\, R_{i,i+1}$$

- we only need to verify $T_i T_j = T_j T_i$ if $|i-j| \geq 2$ - clear.

and $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$: let us take $i=1$ -

- this relation becomes : $(12) R_{12} (23) R_{23} (12) R_{12} = (23) R_{23} (12) R_{12} (23) R_{23}$

$$\equiv \cancel{(12)(23)(12)} \cdot R_{23} R_{13} R_{12} = \cancel{(23)(12)(23)}\, R_{12} R_{13} R_{23}$$

follows by the proposition above. $\square$

4. $\underline{U_\hbar(sl_2)}$ - unital, associative algebra over $\mathbb{C}[[\hbar]]$ -

(Drinfeld-Jimbo) generators : H, E, F

relations : $[H,E] = 2E$; $[H,F] = -2F$.

(recall - $[a,b] = ab - ba$)

$$[E,F] = \frac{q^H - q^{-H}}{q - q^{-1}} = \frac{e^{\frac{\hbar}{2}H} - e^{-\frac{\hbar}{2}H}}{e^{\hbar/2} - e^{-\hbar/2}}$$

$$\left(= H + O(\hbar^2)\right)$$

- $\Delta : \mathcal{U}_\hbar(\mathfrak{sl}_2) \to \mathcal{U}_\hbar(\mathfrak{sl}_2)^{\otimes 2}$:
$$\Delta(H) = H\otimes 1 + 1\otimes H$$
$$\left(K := q^H = e^{\frac{\hbar}{2}H}\right) \qquad \Delta(E) = E\otimes 1 + K\otimes E$$
$$\Delta(F) = F\otimes K^{-1} + 1\otimes F$$

(Proof.- $[H,E] = 2E \quad [H,F] = -2F$ hold clearly.

Check: $\Delta([EF]) = [\Delta(E), \Delta(F)]$:

$$\text{R.H.S.} = [E\otimes 1 + K\otimes E,\ F\otimes K^{-1} + 1\otimes F]$$
$$= [E,F]\otimes K^{-1} + KF\otimes EK^{-1} - FK\otimes K^{-1}E$$
$$+ K\otimes [E,F]$$

– side lemma: $KEK^{-1} = q^2 E$ and $KFK^{-1} = q^{-2}F$.

$\left(KEK^{-1} = Ad(e^{\hbar/2 \cdot H}).E = \exp\left(ad\left(\frac{\hbar}{2}H\right)\right).E\right.$

and $\left. ad\left(\frac{\hbar}{2}H\right).E = \hbar E\right)$.

Continuing with R.H.S. $= \left(\dfrac{K-K^{-1}}{q-q^{-1}}\right)\otimes K^{-1} + K\otimes\left(\dfrac{K-K^{-1}}{q-q^{-1}}\right)$

$$+ KF\otimes K^{-1}E\left(\cancel{q^2 - q^2}\right)^{0}$$

$$= \frac{K\otimes K - K^{-1}\otimes K^{-1}}{q-q^{-1}}$$

$$\text{L.H.S.} = \frac{\Delta(K) - \Delta(K^{-1})}{q-q^{-1}}$$

$$= \frac{K\otimes K - K^{-1}\otimes K^{-1}}{q-q^{-1}}.$$

$$\Delta(H) = H\otimes 1 + 1\otimes H$$
$$\Rightarrow \Delta(K) = e^{\frac{\hbar}{2}\Delta(H)}$$
$$= e^{\frac{\hbar}{2}(H\otimes 1 + 1\otimes H)}$$
$$= e^{\frac{\hbar}{2}(H\otimes 1)} \cdot e^{\frac{\hbar}{2}(1\otimes H)}$$
$$= K\otimes 1 \cdot 1\otimes K = K\otimes K. \ \S$$
).

- Counit : $\varepsilon(H) = \varepsilon(E) = \varepsilon(F) = 0$.

- Antipode : $S : \mathcal{U}_\hbar(sl_2) \longrightarrow \mathcal{U}_\hbar(sl_2)$

$$S(H) = -H \quad (\text{so, } S(K) = K^{-1}.)$$
$$S(E) = -K^{-1}E$$
$$S(F) = -FK$$

(Ex. Verify that S extends to an algebra anti hom.)

$$\Delta(H) = H\otimes 1 + 1\otimes H \underset{\text{mult}\,\circ\, S\otimes 1}{\rightsquigarrow} -H\cdot 1 + 1\cdot H = 0 = \varepsilon(H) \checkmark$$

$$\Delta(E) = E\otimes 1 + K\otimes E \rightsquigarrow -K^{-1}E\cdot 1 + K^{-1}E = 0 = \varepsilon(E) \checkmark$$

$$\Delta(F) = F\otimes K^{-1} + 1\otimes F \rightsquigarrow -FK\cdot K^{-1} + 1\cdot F = 0 = \varepsilon(F) \checkmark$$

5. We will prove next time that $\exists\, \mathcal{R} \in \mathcal{U}_\hbar(sl_2)^{\otimes 2}$ making

$(\mathcal{U}_\hbar(sl_2), \Delta, \varepsilon, S, \mathcal{R})$ = quasi-triangular Hopf algebra.

– also generalize it to any $\mathfrak{g}$ - simple Lie algebra assoc. to a root system.