Lecture 24

0. Recall: $\mathcal{U}_\hbar(sl_2)$ was defined as a Hopf algebra: (over $\mathbb{C}[[\hbar]]$)

- Algebra structure: Generators: $\{H, E, F\}$

Relations: $[H,E] = 2E$ $\quad$ (hence $KE = q^2 EK$, $KF = q^{-2} FK$)

$[H,F] = -2F$

$[E,F] = \dfrac{K - K^{-1}}{q - q^{-1}}$

$$\boxed{q = e^{\frac{\hbar}{2}} \; ; \; K = q^H = e^{\frac{\hbar}{2} \cdot H}}$$

- Coproduct: $\Delta : \mathcal{U}_\hbar(sl_2) \longrightarrow \mathcal{U}_\hbar(sl_2)^{\otimes 2}$ is given by

$$\Delta(H) = H \otimes 1 + 1 \otimes H \quad \left(\text{hence } \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}\right)$$

$$\Delta(E) = E \otimes 1 + K \otimes E$$

$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F$$

- Counit: $\varepsilon : \mathcal{U}_\hbar(sl_2) \longrightarrow \mathbb{C}[[\hbar]]$ is given by

$$\varepsilon(H) = \varepsilon(E) = \varepsilon(F) = 0.$$

(hence, $\varepsilon(K^{\pm 1}) = 1$)

- Antipode: $S : \mathcal{U}_\hbar(sl_2) \to \mathcal{U}_\hbar(sl_2)$ anti-hom. of alg (auto.):

$$S(H) = -H \quad \left(\text{hence } S(K) = K^{-1}.\right)$$

$$S(E) = -K^{-1}E$$

$$S(F) = -FK$$

1. Some representations of $\mathcal{U}_\hbar(sl_2)$. - Let $n \in \mathbb{Z}_{\geq 0}$ and let

$$L_n = \big((n+1)\text{-dim'l } \mathbb{C}\text{-vector space}\big) \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]] \quad \text{- basis } \{v_0, \dots, v_n\}$$

(extend scalars)

$U_h(sl_2)$ acts on L_n by: $\quad H.v_k = (n-2k)v_k$ (as before)

$\boxed{\text{Notation: } \ell\in\mathbb{Z} \rightsquigarrow [\ell] = \frac{q^{\ell}-q^{-\ell}}{q-q^{-1}}}$ $\quad$ (hence $K.v_k = q^{n-2k}.v_k$)

$$E v_k = [n-k+1]_q . v_{k-1} \qquad F.v_k = [k+1]_q v_{k+1}.$$

Proof. $\quad [H,E] = 2E$ and $[H,F] = -2F$ are clear.

$$\begin{aligned}
EF.v_k - FE.v_k &= [k+1]\left(E.v_{k+1}\right) - [n-k+1]\left(F v_{k-1}\right) \\
&= \left([k+1][n-k] - [n-k+1][k]\right) v_k \\
&= \frac{(q^{k+1}-q^{-k-1})(q^{n-k}-q^{-n+k}) - (q^{n-k+1}-q^{-n+k-1})(q^{k}-q^{-k})}{(q-q^{-1})^2} . v_k \\
&= \frac{\cancel{q^{n+1}} - q^{-n+2k+1} - q^{n-2k-1} + \cancel{q^{-n-1}} - \cancel{q^{n+1}} + q^{n-2k+1} + q^{-n+2k-1} - \cancel{q^{-n-1}}}{(q-q^{-1})^2} v_k \\
&= \frac{q^{n-2k}(q-q^{-1}) - q^{-n+2k}(q-q^{-1})}{(q-q^{-1})^2} v_k = [n-2k] . v_k \\
&= \frac{K-K^{-1}}{q-q^{-1}} . v_k
\end{aligned}$$

$\square$

2. Another example. $V = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n\text{-fold}}$. $\left(\underset{\mathbb{C}}{\otimes} \mathbb{C}[[\hbar]]\right)$

Lemma.
$$\Delta^{(n)}(E) = \sum_{j=1}^{n} \underbrace{K \otimes \cdots \otimes K}_{j-1} \otimes \underset{\underset{j^{th}}{\uparrow}}{E} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j},$$

$$\Delta^{(n)}(F) = \sum_{j=1}^{n} \underbrace{1 \otimes \cdots \otimes 1}_{j-1} \otimes \underset{\underset{j^{th}}{\uparrow}}{F} \otimes \underbrace{K^{-1} \otimes \cdots \otimes K^{-1}}_{n-j}, \text{ where}$$

$\Delta^{(n)} : U_\hbar(sl_2) \longrightarrow U_\hbar(sl_2)^{\otimes n}$ $= \Delta$ for $n=2$ and

$\Delta^{(n+1)} = \Delta \otimes id^{\otimes n-1} \circ \Delta^{(n)}$ (for example). Co-associativity

of Δ gives : $\Delta^{(n+1)} = id^{\otimes i} \otimes \Delta \otimes id^{\otimes n-i-1}$ $\forall\ i = 0, \ldots, n-1$.

(Proof of this lemma is trivial!)

Lemma $\Rightarrow$ $U_\hbar(sl_2) \circlearrowright V$ given in the basis $\left\{ |\underline{s}\rangle \;\middle|\; \begin{array}{l}\underline{s} = s_1 \ldots s_n \\ s_i \in \{\uparrow, \downarrow\}\end{array} \right\}$

$$H \cdot |\underline{s}\rangle = \left(\#\{j \mid s_j = \uparrow\} - \#\{j \mid s_j = \downarrow\}\right) |\underline{s}\rangle$$

$$E \cdot |\underline{s}\rangle = \sum_{j=1}^{n} q^{\#\{i<j+1 \,:\, s_i = \uparrow\} - \#\{i<j+1 \,:\, s_i = \downarrow\}} \boxed{|u_j \underline{s}\rangle} \to 0 \text{ if } s_j = \uparrow$$

$\to$ i^{th} entry $= s_i$ $(i \neq j)$
$= \downarrow$ $(i = j$ & $s_j = \uparrow)$.

— similarly for F.

$$\left(\text{e.g. } E \cdot |\downarrow \cdots \downarrow\rangle = \sum_{j=1}^{n} q^{j-1} \cdot |\downarrow \cdots \downarrow \underset{\underset{j^{th}\text{spot}}{\uparrow}}{\uparrow} \downarrow \cdots \downarrow\rangle \quad . \right)$$

3. Universal $\mathcal{R}$-matrix of $U_\hbar(sl_2)$. Following Drinfeld's idea: consider the subalgebras of $\mathcal{U}$ ($= U_\hbar(sl_2)$ from now on.)

$\mathcal{U}^{\geq 0}$: generated by $\{H, E\}$ - relation : $HE = E(H+2)$

$\mathcal{U}^{\leq 0}$: " " $\{H, F\}$ - relation : $HF = F(H-2)$

$$\left.\begin{aligned} \Delta(E) &= E\otimes 1 + K\otimes E \\ \Delta(F) &= F\otimes K^{-1} + 1\otimes F \end{aligned}\right\}$$ $\mathcal{U}^{\geq 0}$ & $\mathcal{U}^{\leq 0}$ are sub-bi-algebras of $\mathcal{U}$.

We consider a pairing $(\cdot\,,\cdot) : \mathcal{U}^{\leq 0} \times \mathcal{U}^{\geq 0} \to \mathbb{C}((\hbar))$ s.t.:

- $(H, H) = \dfrac{2}{\ln(q)}$ ← old pairing $(h,h)=2$; $(F, E) = \dfrac{1}{q - q^{-1}}$

- $(1, x) = \varepsilon(x)$ $\forall\, x \in \mathcal{U}^{\geq 0}$; $(y, 1) = \varepsilon(y)$ $\forall\, y \in \mathcal{U}^{\leq 0}$.

- $(y, x_1 x_2) = (\Delta(y), x_1 \otimes x_2)$ ← (pairing is extended as product of componentwise pairing.)
  $\forall\, y \in \mathcal{U}^{\leq 0}$ & $x_1, x_2 \in \mathcal{U}^{\geq 0}$.

- $(y_1 y_2, x) = (y_1 \otimes y_2, \Delta^{op}(x))$ $\forall\, y_1, y_2 \in \mathcal{U}^{\leq 0}$, $x \in \mathcal{U}^{\geq 0}$.

(such pairings are called Hopf pairing)

$\mathcal{R}$ = canonical tensor of $(\cdot,\cdot)$ satisfies

Cabling Id-s:
$$\left.\begin{aligned} \Delta\otimes \mathrm{Id}\,(\mathcal{R}) &= \mathcal{R}_{13}\,\mathcal{R}_{23} \\ \mathrm{Id}\otimes\Delta\,(\mathcal{R}) &= \mathcal{R}_{13}\,\mathcal{R}_{12} \end{aligned}\right\} \begin{aligned} &\text{in } \mathcal{U}^{\leq 0}\otimes\mathcal{U}^{\leq 0}\otimes\mathcal{U}^{\geq 0}. \\ &\text{in } \mathcal{U}^{\leq 0}\otimes\mathcal{U}^{\geq 0}\otimes\mathcal{U}^{\geq 0}. \end{aligned}$$

4. Computation of $\mathcal{R}$:

Note that $\mathcal{R}$ is defined as follows: let $\{A_K\}$ be a basis of $\mathcal{U}^{\leq 0}$ and $\{B_K\}$ be the basis of $\mathcal{U}^{\geq 0}$ dual to $\{A_K\}$:

$$(A_K, B_\ell) = \delta_{K,\ell}.$$

Then $\mathcal{R} = \sum A_K\otimes B_K \in \mathcal{U}^{\leq 0}\otimes\mathcal{U}^{\geq 0} \subset \mathcal{U}\otimes\mathcal{U}$.

(4.1) Proof of Cabling Id-s. I. As both sides are in $\mathcal{U}^{\leq 0}\otimes\mathcal{U}^{\leq 0}\otimes\mathcal{U}^{\geq 0}$, we pair them with a typical tensor of $\mathcal{U}^{\geq 0}\otimes\mathcal{U}^{\geq 0}\otimes\mathcal{U}^{\leq 0} \ni B_K\otimes B_\ell\otimes A_s$ to get

$$\big(B_K\otimes B_\ell\otimes A_s,\ \Delta\otimes \mathrm{id}\,(\mathcal{R})\big) = \big(B_K\otimes B_\ell,\ \Delta(A_s)\big)$$
$$\big(B_K\otimes B_\ell\otimes A_s,\ R_{13}\cdot R_{23}\big) = \big(A_s,\ B_K B_\ell\big)$$

are equal by our axioms on $(\cdot,\cdot)$ $[A\text{'s}\in\mathcal{U}^{\leq 0};\ B\text{'s}\in\mathcal{U}^{\geq 0}.]$

Similarly the second one.

(4.2). For us: $\mathcal{U}^{\leq 0}$ has ~~the~~ a following basis

$$\{ H^a F^b \; : \; a, b \in \mathbb{Z}_{\geq 0} \}$$

Computation of $\mathcal{R}$ $\leftrightarrow$ basis of $\mathcal{U}^{\geq 0}$ dual to our chosen one for $\mathcal{U}^{\leq 0}$

$$\longleftrightarrow \quad (H^a F^b, H^c E^d) = ?$$

Step 1. $\quad (H, H) = \dfrac{2}{\ln(q)} \quad \rightsquigarrow \quad (H^n, H^m) = \delta_{n,m} \dfrac{n! \, 2^n}{\ln(q)^n}$

Proof: let us write $\dfrac{1}{t} = \dfrac{2}{\ln(q)}$ - so $(H, H) = \dfrac{1}{t}$.

We also know that $(1, H) = \varepsilon(H) = 0 \quad (= (H, 1))$.

$\boxed{\text{Rk: } \Delta(H) = \Delta^{op}(H)}$ Assume $n \geq m$. Then $(H^n, H^m) = (H \otimes H \otimes \ldots \otimes H, \Delta^{(n)}(H^m))$

$$\Delta^{(n)}(H) = \sum_{j=1}^{n} 1^{\otimes (j-1)} \otimes \underset{\substack{\uparrow \\ j^{th} \text{ spot}}}{H} \otimes 1^{\otimes (n-j)} = \sum_{j=1}^{n} H^{(j)} \text{ (notation!)}$$

$$\Delta^{(n)}(H^m) \underset{\substack{\uparrow \\ \text{alg. hom.}}}{=} (\Delta^{(n)}(H))^m = (H^{(1)} + \ldots + H^{(n)})^m.$$

Since $m \leq n$; if $m < n$ any monomial appearing in this expansion will have 1 at some tensor component $\rightsquigarrow \varepsilon(H) = 0$ will give zero once paired with $(\underbrace{H \otimes \ldots \otimes H}_{n}, -)$.

$m = n$ - only monomial that matters is $H^{(1)} H^{(2)} \ldots H^{(n)}$ which appear with coefficient $n!$

$$\Rightarrow \quad (H^n, H^m) = \delta_{m,n} \cdot n! \, (H, H)^n$$

Step 2. $\langle F^n, E^m \rangle = \delta_{m,n}$

(clear – as before – or by weight preservation)

Proof – $\langle F^n, E^m \rangle = \left(F \otimes F^{n-1}, \Delta^{op}(E^m) \right)$

$\Delta^{op}(E^m) = (E \otimes K + 1 \otimes E)^m$ ← coeff of $E \otimes K \cdot E^{m-1}$ is :

$$E \otimes \sum_{j=0}^{m-1} E^j . K . E^{m-1-j} = \left(\sum_{j=0}^{m-1} q^{-2j} \right) E \otimes K . E^{m-1}$$

$$\frac{1 - q^{-2m}}{1 - q^{-2}} = q^{-m+1} \cdot [m]$$

$$\Rightarrow \left(F^n, E^m \right) = \frac{1}{(q - q^{-1})} \cdot \left(F^{n-1}, K \cdot E^{m-1} \right) \cdot q^{-m+1} \cdot [m]$$

- $\left(F^{n-1}, K . E^{m-1} \right) = \left(F^{n-1}, E^{m-1} \right)$ since the left side of this equation becomes $\left(\Delta(F^{n-1}), K \otimes E^{m-1} \right)$

$\Delta(F^{n-1}) = (F \otimes K^{-1} + 1 \otimes F)^{n-1}$ ~ only relevant term to pair with $K \otimes E^{m-1}$ is (by weight reasons) $1 \otimes F^{n-1}$

and $(1, K) = \varepsilon(K) = e^{\frac{h}{2} \cdot 0} = 1$.

Hence $\left(F^n, E^m \right) = \delta_{n,m} \dfrac{1}{(q - q^{-1})^n} \cdot [n]! \, q^{-(n-1) - (n-2) \cdots -1}$

$$= \delta_{n,m} \cdot \frac{[n]! \, q^{-\frac{n(n-1)}{2}}}{(q - q^{-1})^n}$$

Last step: $\langle H^a F^b, H^c E^d \rangle$

$$= \langle H^a, H^c \rangle \cdot \langle F^b, E^d \rangle \qquad \text{(Prove this!)}$$

$$= \delta_{ac}\,\delta_{bd} \cdot \frac{a!\cdot 2^a}{(\ln(q))^a}\;\frac{[b]!\; q^{-\frac{b(b-1)}{2}}}{(q-q^{-1})^b}$$

⇝ Canonical tensor: $$R = \sum_{n,m\geq 0}\left(\frac{\ln(q)^n}{2^n\cdot n!}\, H^n\otimes H^n\right)\left(\frac{(q-q^{-1})^m}{[m]!}\, q^{\frac{m(m-1)}{2}}\, F^m\otimes E^m\right)$$

$$= q^{\frac{H\otimes H}{2}}\cdot \underbrace{\left(\sum_{m\geq 0}\frac{(q-q^{-1})^m}{[m]!}\, q^{\frac{m(m-1)}{2}}\cdot (F\otimes E)^m\right)}$$

Canonical tensor of $\mathfrak{h}\otimes\mathfrak{h}$

$$= q^{\frac{H\otimes H}{2}}\cdot \exp_q\big((q-q^{-1})F\otimes E\big)$$: q-exponential series

$$\exp_q(x) = \sum_{n\geq 0} q^{n(n-1)/2}\,\frac{x^n}{[n]!}$$ – solution of

$$\exp_q(qx) - \exp_q(q^{-1}x) = \sum_{n\geq 1} q^{\frac{n(n-1)}{2}}\,\frac{(q-q^{-1})\,x^n\,\cancel{(q^n-q^{-n})}}{\cancel{(q^n-q^{-n})}\cdot[n-1]!}$$

$$= (q-q^{-1})\,x\cdot\sum_{m\geq 0} q^{(m+1)\frac{m}{2}}\,\frac{x^m}{[m]!} \qquad \left(\frac{(m+1)m}{2} = \frac{(m-1)m}{2}+m\right)$$

$$= (q-q^{-1})\,x\,\exp_q(qx).$$