Lecture 25

0. Recall: $(\mathcal{U}_\hbar(sl_2), \Delta, \varepsilon, S)$ – is a Hopf algebra. $(\mathcal{U} = \mathcal{U}_\hbar(sl_2))$

$$\underset{\{H,F\}}{\overset{\uparrow}{\mathcal{U}^{\leq 0}}}, \quad \underset{\{H,E\}}{\overset{\uparrow}{\mathcal{U}^{\geq 0}}} \subset \mathcal{U} \quad \text{– Hopf subalgebras}$$

(generators)

- Drinfeld pairing $(\cdot,\cdot) : \mathcal{U}^{\leq 0} \times \mathcal{U}^{\geq 0} \longrightarrow \mathbb{C}((\hbar))$ defined by

(i) $(1, x) = \varepsilon(x)$; $(y, 1) = \varepsilon(y)$

(ii) $(y, x_1 x_2) = (\Delta(y), x_1 \otimes x_2)$

$(y_1 y_2, x) = (y_1 \otimes y_2, \Delta^{op}(x))$

(iii) $(H,H) = \dfrac{2}{\ln(q)}$; $(F,E) = \dfrac{1}{q - q^{-1}}$

- Canonical tensor $\mathcal{R} \in \mathcal{U}^{\leq 0} \otimes \mathcal{U}^{\geq 0} \subset \mathcal{U} \otimes \mathcal{U}$, then satisfies

$$\left.\begin{aligned} \Delta \otimes id \, (\mathcal{R}) &= \mathcal{R}_{13} \mathcal{R}_{23} \\ id \otimes \Delta \, (\mathcal{R}) &= \mathcal{R}_{13} \mathcal{R}_{12} \end{aligned}\right] \text{Cabling identities.}$$

- Explicitly: $\mathcal{R} = q^{\frac{H \otimes H}{2}} \cdot \exp_q\left((q - q^{-1}) F \otimes E\right)$

1. Prop. For every $x \in \mathcal{U}$, we have:

$$\Delta^{op}(x) = \mathcal{R} \, \Delta(x) \, \mathcal{R}^{-1}.$$

First proof.- Directly using the explicit formula of $\mathcal{R}$.

$x = H$: the statement is clear since $\mathcal{R}$ is of weight 0

(i.e. $[H\otimes 1 + 1\otimes H, \mathcal{R}] = 0$).

$x = E$: $\mathcal{R}\cdot\Delta(E) = \mathcal{R}\cdot(E\otimes 1 + K\otimes E)$

$\Delta^{op}(E)\cdot\mathcal{R} = (E\otimes K + 1\otimes E)\mathcal{R}$) are equal - To prove.

Claim 1. $q^{-\frac{H\otimes H}{2}}(E\otimes 1)\,q^{\frac{H\otimes H}{2}} = E\otimes K^{-1}$

(Pf. L.H.S. $= \mathrm{Ad}\left(\exp\left(-\frac{\hbar}{4}H\otimes H\right)\right)\cdot E\otimes 1$

$= \exp\left(\mathrm{ad}\left(-\frac{\hbar}{4}H\otimes H\right)\right)\cdot E\otimes 1$. Now

$\mathrm{ad}\left(\frac{-\hbar}{4}H\otimes H\right)\cdot E\otimes 1 = -\frac{\hbar}{2}E\otimes H = \left(1\otimes\left(-\frac{\hbar}{2}H\right)\right)(E\otimes 1)$

$\Rightarrow$ LHS $= E\otimes e^{-\frac{\hbar}{2}H} = E\otimes K^{-1}$ $\square$)

Thus, we have to prove: $\overline{\mathcal{R}}\,(E\otimes 1 + K\otimes E) = q^{-\frac{H\otimes H}{2}}(E\otimes K + 1\otimes E)\,q^{\frac{H\otimes H}{2}}\,\overline{\mathcal{R}}$

$= (E\otimes 1 + K^{-1}\otimes E)\,\overline{\mathcal{R}}$; where

$$\overline{\mathcal{R}} = \exp_q\left((q-q^{-1})F\otimes E\right) = \sum_{n\geq 0} c_n F^n\otimes E^n . \quad \left(c_n = \frac{q^{\frac{n(n-1)}{2}}(q-q^{-1})^n}{[n]!}\right).$$

Comparing terms of equal weight - we need to prove

$$c_n F^n E\otimes E^n + c_{n-1}F^{n-1}K\otimes E^n = c_n EF^n\otimes E^n + c_{n-1}K^{-1}F^{n-1}\otimes E^n$$

$\forall n\geq 0$. Convention $c_{-1} = 0$.

$\equiv$ly $\quad c_n[E, F^n] = c_{n-1}\left(F^{n-1}K - K^{-1}F^{n-1}\right)$

$\Longleftrightarrow$ $\quad [E, F^n] = \frac{[n]\,q^{-n+1}}{q-q^{-1}}\left(F^{n-1}K - K^{-1}F^{n-1}\right)$

Plug-in value of c_n

$$\boxed{[E, F^n] = \frac{[n]}{q - q^{-1}} \left(q^{n-1} K - q^{-n+1} K^{-1} \right) F^{n-1}}$$

(Exercise - prove this identity by induction on n).

$\square$

2. Second proof of the proposition - indirect - using only the axioms of Hopf algebras.

Notation: (Sweedler) - for a Hopf algebra $\mathcal{H}$, and $z \in \mathcal{H}$ -

$$\Delta^{(n)}(z) = \sum_a z_a^{(1)} \otimes z_a^{(2)} \otimes \cdots \otimes z_a^{(n)}$$

$(\in \mathcal{H}^{\otimes n})$ - just written as $z^{(1)} \otimes \cdots \otimes z^{(n)}$ for convenience.

Lemma - For $x \in \mathcal{U}^{\geq 0}$ and $y \in \mathcal{U}^{\leq 0}$ we have the following identity in $\mathcal{U}$:

$$\boxed{xy = y^{(2)} x^{(2)} \cdot \left(S^{-1}(y^{(1)}), x^{(1)} \right) \left(y^{(3)}, x^{(3)} \right)} \quad - (*)$$

Note: L.H.S. $\in \mathcal{U}^{\geq 0} \cdot \mathcal{U}^{\leq 0}$ - so this is a <u>straightening relation</u>

RHS $\in \mathcal{U}^{\leq 0} \cdot \mathcal{U}^{\geq 0}$

Example. - take $x = E$ and $y = F$

$$\Delta^{(3)}(x) = E \otimes 1 \otimes 1 + K \otimes E \otimes 1 + K \otimes K \otimes E$$

$$\Delta^{(3)}(y) = F \otimes K^{-1} \otimes K^{-1} + 1 \otimes F \otimes K^{-1} + 1 \otimes 1 \otimes F$$

Now $S(F) = -FK$, $S(K) = K^{-1}$ $\Rightarrow$ $S^{-1}(F) = -KF$.

So RHS of the equation from the lemma above is:

$$K^{-1}\cdot(-KF, E)(K^{-1}, 1) + FE\cdot(1, K)(K^{-1}, 1) + K\cdot(1, K)(F, E) = \frac{-K^{-1}}{q-q^{-1}} + FE + \frac{K}{q-q^{-1}} \quad ✓$$

$$\left(\text{note: } (K, F, E) = (K\otimes F, \Delta^{op}(E)) = (K\otimes F, 1\otimes E + E\otimes K) = (K,1)\cdot(F,E) = \underbrace{\varepsilon(K)}_{=1}\cdot\frac{1}{q-q^{-1}}.\right)$$

Remark.- One can check, easily, that if (*) from page 3 holds for (x_1, y) and (x_2, y) then it holds for (x_1x_2, y) - and similarly the other way around. Hence it is enough to verify (*) on generators of $\mathcal{U}^{\leq 0}$ & $\mathcal{U}^{\geq 0}$ to prove the lemma. Thus the example we worked out suffices.

3. Recall that $\mathcal{R} \in \mathcal{U}^{\leq 0}\otimes\mathcal{U}^{\geq 0}$ is the canonical tensor -

$$\mathcal{R} = \sum_k A_k \otimes B_k$$

where $\{A_k\}$ is a basis of $\mathcal{U}^{\leq 0}$, $\{B_k\}$ " " " $\mathcal{U}^{\geq 0}$ s.t. $(A_k, B_l) = \delta_{kl}$.

Let $y \in \mathcal{U}^{\leq 0}$ and let us prove that $\mathcal{R}\,\Delta(y) = \Delta^{op}(y)\,\mathcal{R}$.

$$R\cdot\Delta(y) = A_k\cdot y^{(1)}\otimes \underbrace{B_k\, y^{(2)}}_{\substack{\text{"wrong order}\\ B\in\mathcal{U}^{\geq 0};\ y^{(2)}\in\mathcal{U}^{\leq 0}\text{"}}}$$

(summation is not written! $\Delta(y) = y^{(1)}\otimes y^{(2)}$)

We will apply Lemma 2 to straighten the second tensor factor.

For this we need • $\boxed{\Delta^{(3)}(y^{(2)})}$ – $\quad y^{(1)}\otimes\Delta^{(3)}(y^{(2)}) = \Delta^{(4)}(y)$ by defn

$$= y^{(1)}\otimes y^{(2)}\otimes y^{(3)}\otimes y^{(4)}$$ (abusing Sweedler's notation)

• $\boxed{\Delta^{(3)}(B_k)}$ – $\quad A_k\otimes\Delta^{(3)}(B_k)$

$$= (1\otimes\Delta^{(3)})(R) = R_{14}R_{13}R_{12} \quad \text{by Cabling Id 2.}$$

$$= A_i A_j A_k\otimes B_k\otimes B_j\otimes B_i$$

$$\Rightarrow R\cdot\Delta(y) = A_i A_j A_k\cdot y^{(1)}\otimes y^{(3)} B_j\cdot\left(S^{-1}(y_2), B_k\right)\left(y^{(4)}, B_i\right)$$

As $\{A_k\}$ and $\{B_k\}$ are dual to each other, $\forall\, z\in\mathcal{U}^{\leq 0}$

$$z = \sum_i A_i\,(z, B_i)$$

$$\Rightarrow R\cdot\Delta(y) = y^{(4)} A_j \boxed{S^{-1}(y_2)\, y^{(1)}}\otimes y^{(3)} B_j$$

$$= y^{(2)} A_j\otimes y^{(1)} B_j = \Delta^{op}(y)\cdot R. \qquad \square$$

see next page

4. Explanation of the last identity from page 5:

Recall the axiom: $\text{mult} \circ (S \otimes id) \circ \Delta = \varepsilon$ - written in Sweedler's notation : $S(z^{(1)})\, z^{(2)} = \varepsilon(z)$

$\downarrow$ Apply S^{-1} - remember it is anti-hom.

$$S^{-1}(z^{(2)})\, z^{(1)} = \varepsilon(z)$$

Now $y^{(1)} \otimes y^{(2)} \otimes y^{(3)} \otimes y^{(4)} = \Delta^{(4)}(y)$ can be written as

$$\Delta \otimes id \otimes id\,(\Delta \otimes id\,(\Delta(y))) \rightsquigarrow S^{-1}(y^{(2)})\, y^{(1)} \otimes y^{(3)} \otimes y^{(4)}$$

$$= (\varepsilon \otimes id \otimes id)(\Delta \otimes id\,(\Delta(y))) = 1 \otimes \Delta(y) = 1 \otimes y^{(1)} \otimes y^{(2)}.$$

~~5. Weyl group action - (compare with Lecture 15).~~

~~Definition (Luszting)~~

$$\cancel{\mathcal{S} = \exp_{q^{-1}}(q^{-1} E K^{-1})\, \exp_{q^{-1}}(-F)\, \exp_{q^{-1}}(q E K)\, q^{\frac{H(H+1)}{2}}}$$

~~(q-analogue of the classical - $s = \exp(e)\exp(-f)\exp(e)$.)~~

~~- As before, $\mathcal{S}$ makes sense on any $U_h(sl_2)$-representation where E & F act locally nilpotently.~~

~~Let $\lambda \in \mathbb{Z}_{\geq 0}$ and let L_λ be the irreducible $(\lambda+1)$-dim'l representation of $U_h(sl_2)$. It has a basis $\{v_0, \dots, v_\lambda\}$~~