Lecture 26. - Quantum groups $U_\hbar(\mathfrak{g})$.

0. Recall - the root data is encoded in its Cartan matrix $A = (a_{ij})_{i,j \in I}$

(i.e. $a_{ii} = 2$ $(\forall i)$; $a_{ij} \in \mathbb{Z}_{\leq 0}$ $\forall i \neq j$; $\exists$ diagonal $D = (d_i)_{i \in I}$; $d_i \in \mathbb{Z}_{>0}$

s.t. DA is symmetric and positive-definite -

— classified by Dynkin diagrams.)

1. $\mathfrak{h} = \mathbb{C}$ - span of h_i $(i \in I)$. $\alpha_j \in \mathfrak{h}^*$ defined by $\alpha_j(h_i) = a_{ij}$.
$\omega_j \in \mathfrak{h}^*$ defined by $\omega_j(h_i) = \delta_{ij}$

Root lattice

$$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}^* \quad ; \quad Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$$

Weight lattice $P = \bigoplus_{i \in I} \mathbb{Z}\omega_i \subset \mathfrak{h}^*$. Note: $\alpha_j = \underbrace{\sum_i a_{ij}\,\omega_i}_{\in P}$

$\Rightarrow$ $Q \subset P$.

$R \subset \mathfrak{h}^* \setminus \{0\}$ - corresponding root system

($R = W$-orbit of $\{\alpha_i : i \in I\}$

$W \longrightarrow \langle s_i \ldots \rangle$ $\boxed{s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \text{ action}}$

$$\left.\begin{array}{l} R_+ = R \cap Q_+ \\ R_- = -R_+ \end{array}\right\} \quad R = \underset{\substack{\uparrow \\ \text{positive roots}}}{R_+} \sqcup R_- \quad .)$$

Non-degenerate form on $\mathfrak{h}^*$: $(\alpha_i, \alpha_j) = d_i a_{ij}$

$\rightsquigarrow$ non-degenerate form on $\mathfrak{h}$: $(h_i, h_j) = \frac{1}{d_j} a_{ij}$

$(\alpha_i \longleftrightarrow d_i h_i)$

2. We define $\widetilde{\mathcal{U}}_\hbar(\mathfrak{g})$ - to be unital associative (completed!) algebra over $\mathbb{C}[[\hbar]]$: generators: $\mathfrak{h}$; $\{E_i, F_i\}_{i \in I}$

relations $[h, h'] = 0 \quad \forall h \in \mathfrak{h}$.

$[h, E_i] = \alpha_i(h) E_i$; $[h, F_j] = -\alpha_j(h) F_j$

$$\boxed{[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}}$$ - where

$q = e^{\hbar/2}$; $q_i = q^{d_i}$; $K_i = q_i^{H_i}$

(i.e. $\{H_i = h_i, E_i, F_i\}$ - generate a copy of $\mathcal{U}_{d_i\hbar}(\mathfrak{sl}_2)$.)

- Coproduct and antipode are defined so that each $\mathcal{U}_{d_i\hbar}(\mathfrak{sl}_2)$ is a Hopf subalgebra:

$\Delta(x) = x \otimes 1 + 1 \otimes x \quad \forall x \in \mathfrak{h}$.

$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$

$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$

$S(E_i) = -K_i^{-1} E_i$

$S(F_i) = -F_i K_i$

$\varepsilon(H) = 0 = \varepsilon(E_i) = \varepsilon(F_i)$

(We are going to mimick our construction of $\mathfrak{g}(A)$ -)

3. $\widetilde{\mathcal{U}}_\hbar^{\leq 0}$ and $\widetilde{\mathcal{U}}_\hbar^{\geq 0}$ - are defined to be the subalgebras of $\widetilde{\mathcal{U}}_\hbar$ generated by $\{H \in \mathfrak{h} ; F_i \ (i \in I)\}$ and $\{H \in \mathfrak{h} ; E_i \ (i \in I)\}$

<u>Prop.</u> There is a unique bilinear form ($\underline{\text{NOT}}$ non-degenerate!)

$$\widetilde{\mathcal{U}}_\hbar^{\leq 0} \times \widetilde{\mathcal{U}}_\hbar^{\geq 0} \longrightarrow \mathbb{C}((\hbar))$$

$$\forall H, H' \in \mathfrak{h} : \quad (H, H')_\hbar = \frac{(H,H')}{\ln(q)} = \frac{2}{\hbar}(H,H')$$

$$\forall i,j \in I \quad (F_i, E_j) = \frac{\delta_{ij}}{q_i - q_i^{-1}} = \delta_{ij}\left(\frac{1}{\hbar \cdot d_i} + \cdots\right)$$

$$(y, x_1 x_2) = (\Delta(y), x_1 \otimes x_2)$$

$$(y_1 y_2, x) = (y_1 \otimes y_2, \Delta^{op}(x))$$

Involution $\theta : \widetilde{\mathcal{U}}_\hbar \circlearrowleft \ E_i \leftrightarrow F_i \quad H \mapsto H$ - then $\begin{pmatrix} \theta(ab) = \theta(b)\theta(a) \\ \Delta(\theta(a)) \\ = \theta \otimes \theta(\Delta^{op}(a)) \end{pmatrix}$

$$(y, x) = (\theta(x), \theta(y))$$

Grading $\widetilde{\mathcal{U}}^{\geq 0 / \leq 0}$ - are graded by Q_+ : $\deg(E_i) = \alpha_i$

as such $\widetilde{\mathcal{U}}^{\geq 0/\leq 0} = \underbrace{(\mathrm{Sym}_\hbar \mathfrak{h})}_{\widetilde{\mathcal{U}}^0} \cdot \underbrace{\boxed{\text{free assoc. algebra in } E_i / F_i \ (i \in I)}}_{\widetilde{\mathcal{U}}^{+/-}}$

$$\tilde{\mathcal{U}}^{+/-} = \bigoplus_{\alpha \in Q_+} \tilde{\mathcal{U}}^{+}_{\pm\alpha} = \mathbb{C}\langle E_i / F_i \ (i \in I)\rangle .$$

4. Idea: $r^{\pm} \subset \tilde{\mathcal{U}}^{+/-}$ defined by $\{x \mid (x, y) = 0 \ \forall y\}$

radical

- $\tilde{\mathcal{U}}^{0} \cdot \tilde{\mathcal{U}}^{-} \times \tilde{\mathcal{U}}^{0} \tilde{\mathcal{U}}^{+} \longrightarrow \mathbb{C}((\hbar))$ factors

  $(p_1 \cdot y, p_2 \cdot x) = (p_1, p_2) \cdot (y, x)$ - Exercise

- $\tilde{\mathcal{U}}^{0} \times \tilde{\mathcal{U}}^{0} = \mathrm{Sym}\,\mathfrak{h} \times \mathrm{Sym}\,\mathfrak{h} \longrightarrow \mathbb{C}((\hbar))$

  if $\{x_i : i \in I\}$ is an orthonormal basis of $\boxed{(H_i H_j) = \frac{1}{d_j} a_{ij}}$

  then $(\underline{x}^{\underline{n}}, \underline{x}^{\underline{m}})_{\hbar} = \dfrac{\delta_{\underline{n},\underline{m}} \prod_{i \in I} n_i!}{\ln(q)^{\sum_i n_i}}$ - non-degenerate

  $\prod_i x_i^{n_i}$ $\qquad$ $\prod_i x_i^{m_i}$ $\qquad$ (same calculation as for $\mathfrak{sl}_2$)

  $\rightsquigarrow$ Canonical tensor $\boxed{\mathcal{R}^0 = q^{\sum_i x_i \otimes x_i}}$.

$\Rightarrow \mathrm{rad}^{\geq 0/\leq 0} \subset \tilde{\mathcal{U}}_{\hbar}^{\geq 0/\leq 0}$ - Hopf ideals by defining property of $(\cdot,\cdot)_{\hbar}$

are of the form $\tilde{\mathcal{U}}^{0} \cdot r^{\pm}$. Let us focus on + case -

$\rightarrow$ $\boxed{\text{To prove: } \tilde{\mathcal{U}}^{-} \cdot \tilde{\mathcal{U}}^{0} \cdot r^{+} \subset \tilde{\mathcal{U}} \text{ is a 2-sided ideal}}$.

5. $K_i = q_i^{h_i} = K_{\alpha_i}$. For $\mu \in Q_+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$,

$$\mu = \sum_{i\in I} \mu_i \alpha_i \rightsquigarrow K_\mu = \prod_{i\in I} K_i^{\mu_i}$$

$\tilde{\mathcal{U}}^{\pm}_{\pm\mu}$ = $\mathbb{C}$-span of monomials in $\{E_i : i \in I\}$ s.t. degree of $E_i = \mu_i$

$$q^h x_\mu q^{-h} = q^{\mu(h)} \cdot x_\mu \quad \forall\, x_\mu \in \mathcal{U}^+_\mu$$

- $(q^h, q^{h'}) = q^{(h,h')}$ $\qquad \varepsilon(q^h) = 1$

$\qquad \Delta(q^h) = q^h \otimes q^h$

6. $\Delta(E_i) = E_i \otimes 1 + \boxed{q_i^{H_i}} \otimes E_i$ K_i

$\Delta^{op}(E_i) = E_i \otimes K_i + 1 \otimes E_i$

$$\Rightarrow \forall\, x_\mu \in \tilde{\mathcal{U}}^+_\mu : \quad \Delta^{op}(x_\mu) = x_\mu \otimes K_\mu + \sum_i r_i'(x_\mu) \otimes E_i K_{\mu-\alpha_i} + \dots + \sum_i E_i \otimes r_i(x_\mu)\cdot K_i + 1 \otimes x_\mu.$$

Definition of $r_i, r_i' : \tilde{\mathcal{U}}^+_\mu \to \tilde{\mathcal{U}}^+_{\mu-\alpha_i}$

Prop. – (1) $r_i(E_j) = r_i'(E_j) = \delta_{ij} \quad \forall\, i,j \in I$.

(2) $r_i(x_1 x_2) = q^{(\mu_2,\alpha_i)} r_i(x_1) x_2 + x_1 r_i(x_2)$

$r_i'(x_1 x_2) = r_i'(x_1) x_2 + q^{(\mu_1,\alpha_i)} x_1 r_i'(x_2)$

$\forall\, x_j \in \tilde{\mathcal{U}}^+_{\mu_j}$; $j = 1,2$.

(3) $\forall\, x \in \widetilde{\mathcal{U}}^+$ and $i \in I$:

$$[x, f_i] = \frac{r_i(x) K_i - K_i^{-1} r_i'(x)}{q_i - q_i^{-1}}$$

Proof. – (1) is obvious. For (2) –

$$\Delta^{op}(x_1 x_2) = x_1 x_2 \otimes K_{\mu_1+\mu_2} + \sum_i r_i'(x_1 x_2) \otimes E_i K_{\mu_1+\mu_2-\alpha_i}$$

$$+ \dots + \sum_i E_i \otimes r_i(x_1 x_2) K_i + 1 \otimes x_1 x_2$$

$$= \Delta^{op}(x_1)\Delta^{op}(x_2) = \Big(x_1 \otimes K_{\mu_1} + \sum_i r_i'(x_1) \otimes E_i K_{\mu_1-\alpha_i} + \dots + \sum_i E_i \otimes r_i(x_1) K_i$$

$$+ 1 \otimes x_1\Big) \cdot \text{(same exp. for 2.)}$$

$$\Rightarrow r_i'(x_1 x_2) = \text{Coeff of } - \otimes E_i K_{\mu_1+\mu_2-\alpha_i} \text{ in}$$

$$r_i'(x_1) x_2 \otimes E_i K_{\mu_1-\alpha_i} K_{\mu_2}$$

$$+ x_1 r_i'(x_2) \otimes K_{\mu_1} E_i K_{\mu_2-\alpha_i}$$

$$q^{(\mu_1,\alpha_i)}$$

Hence, (2) follows.

(3): Obvious for $x = E_j$ $(j \in I)$ due to (1). Take x_1, x_2 as in (2):

$$[x_1 x_2, F_i] = [x_1, F_i] x_2 + x_1 [x_2 F_i]$$

$$= \frac{\left(r_i(x_1) K_i - K_i^{-1} r_i'(x_1)\right) x_2 + x_1 \left(r_i(x_2) K_i - K_i^{-1} r_i'(x_2)\right)}{q_i - q_i^{-1}}$$

$$= \frac{\left(q^{(\mu_2,\alpha_i)} r_i(x_1)x_2 + x_1 r_i(x_2)\right)K_i - K_i^{-1}\left(r_i'(x_1)x_2 + q^{(\mu_1,\alpha_i)} x_1 r_i'(x_2)\right)}{q_i - q_i^{-1}}$$

$$= \frac{r_i(x_1x_2)K_i - K_i^{-1} r_i'(x_1x_2)}{q_i - q_i^{-1}} \qquad \square$$

7. Let $y \in \widetilde{\mathcal{U}}^-_{-(\mu-\alpha_i)}$ and $x \in \widetilde{\mathcal{U}}^+_\mu$.

Prop. – $(F_i y, x) = (F_i, E_i)\cdot \underbrace{(y, r_i(x)\cdot K_i)}_{\text{(non-zero scalar multiple of } (y, r_i(x)))}$

$$(yF_i, x) = (y, r_i'(x))\cdot(F_i, E_iK_{\mu-\alpha_i})$$

Proof. $(yF_i, x) = (y\otimes F_i, \underbrace{\Delta^{op}(x)}_{\text{only relevant term } r_i'(x)\otimes E_iK_{\mu-\alpha_i}})$

$$= (y, r_i'(x))\cdot(F_i, E_iK_{\mu-\alpha_i})$$

$$\left[\,(F_i, E_iK_{\mu-\alpha_i}) = \left((F_i\otimes K_i^{-1} + 1\otimes F_i), E_i\otimes K_{\mu-\alpha_i}\right) = \frac{1}{q_i - q_i^{-1}}\, q^{-(\alpha_i,\mu-\alpha_i)}\,.\right] \qquad \square$$

Cor. – $\mathrm{rad}^+ \subset \widetilde{\mathcal{U}}^+$ is stable under r_i, r_i'. For $x \in \widetilde{\mathcal{U}}^+_\mu$

$$x \in \mathrm{rad}^+_\mu \iff r_i(x), r_i'(x) \in \mathrm{rad}^+_{\mu-\alpha_i} \quad \forall i \in I.$$

8. Propositions 6 & 7 above prove that $\tilde{\mathcal{U}}^{\leq 0} \cdot \mathrm{rad}^+ \subset \tilde{\mathcal{U}}_\hbar(\mathfrak{g})$ is a two-sided ideal. (same for rad^-).

Thus the triangular form $\tilde{\mathcal{U}}^{\leq 0} \cdot \tilde{\mathcal{U}}^{\geq 0} = \tilde{\mathcal{U}}$ is preserved under quotienting by radical of $(\cdot\,,\cdot)$

$$\mathcal{U}_\hbar(\mathfrak{g}) = \tilde{\mathcal{U}}_\hbar(\mathfrak{g}) \Big/ \text{Radical} = \mathcal{U}^- \cdot \mathcal{U}^0 \cdot \mathcal{U}^+ .$$

<u>Prop</u> Let $i \neq j$; $m = 1 - a_{ij}$ and

$$\theta_{ij} := \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} E_i^{m-s} E_j E_i^s \in \tilde{\mathcal{U}}^+_{m\alpha_i + \alpha_j}$$

$$\left(\begin{bmatrix} m \\ n \end{bmatrix}_{q_i} = \frac{[m]_{q_i} [m-1]_{q_i} \cdots [m-n+1]_{q_i}}{[n]_{q_i}!} \ ; \ [\ell]_{q_i} = \frac{q_i^{\ell} - q_i^{-\ell}}{q_i - q_i^{-1}} . \right)$$

Then $r_k(\theta_{ij}) = 0 = r_k'(\theta_{ij}) \quad \forall\, k \in I$.

In particular $\theta_{ij} \in \mathrm{rad}^+ \ \forall\, i \neq j \in I$.

Proof – Let us show it for r_k'. If $k \neq i, j$; $r_k'(\theta_{ij})$ is clearly 0.

$$k = j: \quad r_k'(\theta_{ij}) = \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} q^{(m-s)\boxed{(\alpha_i,\alpha_j)}} E_i^m$$

$d_i a_{ij} = d_i(1-m)$

$$= \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} q_i^{(m-s)(1-m)}$$

$$= q_i^{-m(m-1)} \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} q_i^{s(m-1)} = 0 \text{ by induction on } m$$

[use the identity $\begin{bmatrix} m \\ \ell \end{bmatrix} = q^{-\ell} \begin{bmatrix} m-1 \\ \ell \end{bmatrix} + q^{m-\ell} \begin{bmatrix} m-1 \\ \ell-1 \end{bmatrix}$ – (check this) ;

to get
$$\sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix} q^{s(m-1)} = \sum_{s=0}^{m-1} (-1)^s \begin{bmatrix} m-1 \\ s \end{bmatrix} q^{s(m-2)} - q^{2m\pm 2} \sum_{t=0}^{m-1} (-1)^t \begin{bmatrix} m-1 \\ t \end{bmatrix} q^{t(m-2)} .$$
]

$K=i$:
$$r_i'(E_i^a E_j E_i^b) = r_i'(E_i^a) E_j E_i^b + q_i^{2a+a_{ij}} E_i^a E_j r_i'(E_i^b)$$

$$\boxed{r_i'(E_i^a) = q_i^{a-1} [a]_{q_i} E_i^{a-1}}$$

$$= q_i^{a-1} [a]_{q_i} E_i^{a-1} E_j E_i^b + q_i^{2a+a_{ij}+b-1} [b]_{q_i} E_i^a E_j E_i^{b-1}$$

$\Rightarrow$ Coefficient of $E_i^{m-s-1} E_j E_i^s$ in $r_i'(\theta_{ij}) =$

$$= (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_i q_i^{m-s-1} [m-s]_i + (-1)^{s+1} \begin{bmatrix} m \\ s+1 \end{bmatrix}_i q_i^{2(m-s-1)+a_{ij}+s} [s+1]_i$$

$$= (-1)^s \cdot \frac{[m]_i \cdots [m-s]_i}{[s]_i!} \underbrace{\left(q_i^{m-s-1} - q_i^{2m-2-s+\underbrace{1-m}_{a_{ij}}} \right)}_{0} \qquad \square$$